Note: Heating equation in gyrokinetic system

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1 Gyrokinetic equation

Gyrokinetic equation is given with the gyrokinetic distribution function $h_s(\boldsymbol{R}_s, \boldsymbol{v}, t)$ by

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \frac{1}{B_0}\{\langle\chi\rangle_{\boldsymbol{R}_s}, h_s\} + \frac{1}{B_0}\{\langle\chi\rangle_{\boldsymbol{R}_s}, f_{0s}\} = \frac{q_s f_{0s}}{T_{0s}} \frac{\partial \langle\chi\rangle_{\boldsymbol{R}_s}}{\partial t} + \langle C(h_s)\rangle_{\boldsymbol{R}_s}, \tag{1}$$

where $\{a, b\} = \partial_x a \partial_y b - \partial_y a \partial_x b$ is the Poisson bracket, $\chi = \phi - \boldsymbol{v}\cdot\boldsymbol{A}$.

$$\boldsymbol{R}_s = \boldsymbol{r} + \frac{\boldsymbol{v}\times\hat{\boldsymbol{z}}}{\Omega_s} \tag{2}$$

The equilibrium and perturbed distribution functions are give by

$$\begin{aligned} \delta f_{1s} &= -\frac{q_s\phi}{T_{0s}} f_{0s} + h_s, \\ f_{0s} &= \frac{n_{0s}}{\pi^{3/2} v_{\mathrm{th},s}^3} \exp(-v^2/v_{\mathrm{th},s}^2), \end{aligned} \tag{3}$$

where f_{0s} is Maxwellian. If the Maxwellian has inhomogeneity in the x direction, the fourth term on LHS is

$$\frac{1}{B_0}\{\langle\chi\rangle_{\boldsymbol{R}_s}, f_{0s}\} = -\frac{1}{B_0}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial y}\frac{\partial f_{0s}(x)}{\partial x} = ... = -\frac{f_{0s}}{B_0}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial y}\left(\frac{n_{0s}'}{n_{0s}} + \left(\frac{v^2}{v_{\mathrm{th},s}^2} - \frac{3}{2}\right)\frac{T_{0s}'}{T_{0s}}\right) \tag{4}$$

1.1 Continuity equation

1.2 Momentum equation

1.3 Entropy balance equation

Multiplying (1) by $T_{0s}h_s/f_{0s}$ and integrating over the phase space ,

$$\begin{aligned} &\frac{1}{V}\int\int \frac{T_{0s}h_s}{f_{0s}}\frac{\partial h_s}{\partial t}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s + \frac{1}{V}\int\int \frac{T_{0s}h_s}{f_{0s}}\frac{\partial h_s}{\partial z}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s + \frac{1}{V}\int\int \frac{T_{0s}h_s}{f_{0s}}\frac{1}{B_0}\{\langle\chi\rangle_{\boldsymbol{R}_s}, h_s\}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s \\ &\quad - \frac{1}{V}\int\int \frac{T_{0s}h_s}{f_{0s}}\frac{f_{0s}}{B_0}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial y}\left(\frac{n_{0s}'}{n_{0s}} + \left(\frac{v^2}{v_{\mathrm{th},s}^2} - \frac{3}{2}\right)\frac{T_{0s}'}{T_{0s}}\right)\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s \\ &\quad = \frac{1}{V}\int\int \frac{T_{0s}h_s}{f_{0s}}\frac{q_s f_{0s}}{T_{0s}}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial t}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s + \frac{1}{V}\int\int \frac{T_{0s}h_s}{f_{0s}}\langle C(h_s)\rangle_{\boldsymbol{R}_s}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s \end{aligned} \tag{5}$$

(calculation result for each term is written here)...

$$
\begin{aligned}
\frac{1}{V}\int\int \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{T_{0s}}{2f_{0s}}h_s^2\right)\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s &= \frac{1}{V}\int\int q_s h_s \frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial t}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s \\
&- \frac{1}{V}\int\int \frac{T_{0s}}{L_{n_{0s}}}\frac{h_s}{B_0}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial y}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s - \frac{1}{V}\int\int \frac{T_{0s}}{L_{T_{0s}}}\frac{h_s}{B_0}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial y}\left(\frac{v^2}{v_{\mathrm{th},s}^2}-\frac{3}{2}\right)\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s \\
&+ \frac{1}{V}\int\int \frac{T_{0s}h_s}{f_{0s}}\langle C(h_s)\rangle_{\boldsymbol{R}_s}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s \qquad (6)\\
&= \frac{1}{V}\int\int q_s h_s \frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial t}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s + (I_s - D_s) \qquad (7)
\end{aligned}
$$

where the injection and dissipation terms are defined as

$$
\begin{aligned}
I_s &\equiv -\frac{1}{V}\int\int \frac{T_{0s}}{L_{n_{0s}}}\frac{h_s}{B_0}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial y}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s - \frac{1}{V}\int\int \frac{T_{0s}}{L_{T_{0s}}}\frac{h_s}{B_0}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial y}\left(\frac{v^2}{v_{\mathrm{th},s}^2}-\frac{3}{2}\right)\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s, \qquad (8)\\
D_s &\equiv -\frac{1}{V}\int\int \frac{T_{0s}h_s}{f_{0s}}\langle C(h_s)\rangle_{\boldsymbol{R}_s}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s \geq 0, \qquad (9)
\end{aligned}
$$

where $V = L_x L_y$ with L_x and L_y being the box sizes in the x and y directions, respectively. Note that the dissipation term is positive definite.

2 Fokker–Planck equation

Fokker–Planck equation is given by

$$
\frac{\partial f_s}{\partial t} + \boldsymbol{v}\cdot\nabla f_s + \frac{q_s}{m_s}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f_s}{\partial \boldsymbol{v}} = \left(\frac{\partial f_s}{\partial t}\right)_{\mathrm{coll}}, \qquad (10)
$$

where $f_s(\boldsymbol{r},\boldsymbol{v},t)$ is the velocity distribution function with $s = \mathrm{e},\mathrm{i}$ being species.

Moments of f_s are defined as follow:

$$
\begin{aligned}
n_s &= \int f_s \mathrm{d}\boldsymbol{v}, \qquad (11)\\
\boldsymbol{\Gamma}_s &= n_s\boldsymbol{u}_s = \int \boldsymbol{v} f_s \mathrm{d}\boldsymbol{v}, \qquad (12)\\
\frac{3}{2}P_s &= \int \frac{1}{2}m_s v^2 f_s \mathrm{d}\boldsymbol{v}, \qquad (13)\\
\boldsymbol{Q}_s &= \int \frac{1}{2}m_s v^2 \boldsymbol{v} f_s \mathrm{d}\boldsymbol{v}, \qquad (14)
\end{aligned}
$$

where n_s and $\boldsymbol{\Gamma}_s$ are the density and the particle flux, respectively. $\boldsymbol{u}_s$ is the mean velocity, P_s is the isotropic pressure, and $\boldsymbol{Q}_s$ is the energy flux.

2.1 Continuity equation

Taking the zeroth moment of (10),

$$
\int \frac{\partial f_s}{\partial t}\mathrm{d}\boldsymbol{v} + \int \boldsymbol{v}\cdot\nabla f_s \mathrm{d}\boldsymbol{v} + \int \frac{q_s}{m_s}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v} = \int \left(\frac{\partial f_s}{\partial t}\right)_{\mathrm{coll}}\mathrm{d}\boldsymbol{v}. \qquad (15)
$$

The first term of LHS

$$
\int \frac{\partial f_s}{\partial t}\mathrm{d}\boldsymbol{v} = \frac{\partial n_s}{\partial t} \qquad (16)
$$

The second term of LHS

$$\int \boldsymbol{v}\cdot\nabla f_s \mathrm{d}\boldsymbol{v} = \nabla\cdot\int \boldsymbol{v} f_s \mathrm{d}\boldsymbol{v} = \nabla\cdot\boldsymbol{\Gamma}_s \tag{17}$$

The third term of LHS

$$\int \frac{q_s}{m_s}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v} = \int \frac{q_s}{m_s}\boldsymbol{E}\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v} + \int \frac{q_s}{m_s}\boldsymbol{v}\times\boldsymbol{B}\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v}, \tag{18}$$

The $\boldsymbol{E}$ term is written as follows:

$$\int \frac{q_s}{m_s}\boldsymbol{E}\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v} = \frac{q_s}{m_s}\int \frac{\partial}{\partial \boldsymbol{v}}\cdot(f_s\boldsymbol{E})\mathrm{d}\boldsymbol{v} = \int_{S_\infty} f_s\boldsymbol{E}\cdot d\boldsymbol{S} = 0 \tag{19}$$

On the surface at $|v|\to\infty$, $f_s\to 0$ is required, so $\int_{S_\infty} f_s\boldsymbol{E}\cdot d\boldsymbol{S}=0$. The $\boldsymbol{v}\times\boldsymbol{B}$ terms also vanishes as follow:

$$\int \frac{q_s}{m_s}\boldsymbol{v}\times\boldsymbol{B}\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v} = \frac{q_s}{m_s}\left(\int \frac{\partial}{\partial \boldsymbol{v}}\cdot(f_s\boldsymbol{v}\times\boldsymbol{B})\mathrm{d}\boldsymbol{v} - \int f_s\frac{\partial}{\partial \boldsymbol{v}}\cdot(\boldsymbol{v}\times\boldsymbol{B})\mathrm{d}\boldsymbol{v}\right) \tag{20}$$

$$= \frac{q_s}{m_s}\left(\int_{S_\infty} f_s\boldsymbol{v}\times\boldsymbol{B}\cdot d\boldsymbol{S} - \int f_s\boldsymbol{B}\cdot\left(\frac{\partial}{\partial \boldsymbol{v}}\times\boldsymbol{v}\right)\mathrm{d}\boldsymbol{v}\right) = 0 \tag{21}$$

Since $\partial/\partial\boldsymbol{v}$ is parallel to $\boldsymbol{v}$, the $\partial/\partial\boldsymbol{v}\times\boldsymbol{v}$ term vanishes.

The number of particle do not change due to collisions, so the RHS is 0 (Conservation of particles).

$$\int \left(\frac{\partial f_s}{\partial t}\right)_{\text{coll}} \mathrm{d}\boldsymbol{v} = 0. \tag{22}$$

Thus, the continuity equation is given by

$$\frac{\partial n_s}{\partial t} + \nabla\cdot\boldsymbol{\Gamma_s} = 0. \tag{23}$$

2.2 Momentum equation

Taking the first moment of (10)...

2.3 Transport equation

Taking the second moment of (10),

$$\int \frac{m_s v^2}{2}\frac{\partial f_s}{\partial t}\mathrm{d}\boldsymbol{v} + \int \frac{m_s v^2}{2}\boldsymbol{v}\cdot\nabla f_s\mathrm{d}\boldsymbol{v} + \int \frac{m_s v^2}{2}\frac{q_s}{m_s}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v} = \int \frac{m_s v^2}{2}\left(\frac{\partial f_s}{\partial t}\right)_{\text{coll}}\mathrm{d}\boldsymbol{v}. \tag{24}$$

The first term of LHS is

$$\int \frac{m_s v^2}{2}\frac{\partial f_s}{\partial t}\mathrm{d}\boldsymbol{v} = \frac{\partial}{\partial t}\int \frac{1}{2}m_s v^2 f_s\mathrm{d}\boldsymbol{v} = \frac{3}{2}\frac{\partial}{\partial t}(P_s) \tag{25}$$

The second term of LHS is

$$\int \frac{m_s v^2}{2}\boldsymbol{v}\cdot\nabla f_s\mathrm{d}\boldsymbol{v} = \nabla\cdot\int \frac{1}{2}m_s v^2\boldsymbol{v} f_s\mathrm{d}\boldsymbol{v} = \nabla\cdot\boldsymbol{Q}_s \tag{26}$$

The third term is

$$\int \frac{m_s v^2}{2}\frac{q_s}{m_s}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v} = \frac{q_s}{2}\left(\int v^2\boldsymbol{E}\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v} + \int v^2(\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v}\right). \tag{27}$$

The $\boldsymbol{v} \times \boldsymbol{B}$ term vanishes as follow:

$$\begin{aligned}\int v^2(\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v} &= \int \frac{\partial}{\partial \boldsymbol{v}}\cdot(v^2 f_s \boldsymbol{v}\times\boldsymbol{B})\mathrm{d}\boldsymbol{v} - 2\int \boldsymbol{v}\cdot(f_s\boldsymbol{v}\times\boldsymbol{B})\mathrm{d}\boldsymbol{v} - \int v^2 f_s\frac{\partial}{\partial \boldsymbol{v}}\cdot(\boldsymbol{v}\times\boldsymbol{B})\mathrm{d}\boldsymbol{v} \\ &= \int_{S_\infty}(v^2 f_s\boldsymbol{v}\times\boldsymbol{B})\cdot d\boldsymbol{S} - 2\int \boldsymbol{B}\cdot(f_s\boldsymbol{v}\times\boldsymbol{v})\mathrm{d}\boldsymbol{v} - \int v^2 f_s\boldsymbol{v}\cdot(\frac{\partial}{\partial \boldsymbol{v}}\times\boldsymbol{v})\mathrm{d}\boldsymbol{v} \\ &= 0 \end{aligned} \tag{28}$$

The $\boldsymbol{E}$ term is represented with $\boldsymbol{E} = -\nabla\phi - \partial\boldsymbol{A}/\partial t$ by

$$\begin{aligned}\int v^2\boldsymbol{E}\cdot\frac{\partial f_s}{\partial \boldsymbol{v}}\mathrm{d}\boldsymbol{v} &= \int\frac{\partial}{\partial \boldsymbol{v}}\cdot(v^2 f_s\boldsymbol{E})\mathrm{d}\boldsymbol{v} - 2\int f_s\boldsymbol{v}\cdot\boldsymbol{E}\mathrm{d}\boldsymbol{v} = -2\int f_s\boldsymbol{v}\cdot(-\nabla\phi - \partial\boldsymbol{A}/\partial t)\mathrm{d}\boldsymbol{v} \\ &= 2\left(\int f_s\boldsymbol{v}\cdot\nabla\phi\mathrm{d}\boldsymbol{v} + \frac{\partial}{\partial t}\int f_s\boldsymbol{v}\cdot\boldsymbol{A}\mathrm{d}\boldsymbol{v}\right)\end{aligned} \tag{29}$$

The second moment of the collision term is now

$$\int\frac{m_s v^2}{2}\left(\frac{\partial f_s}{\partial t}\right)_{\mathrm{coll}}\mathrm{d}\boldsymbol{v} = 0 \tag{30}$$

...

Thus, the transport equation is given by

$$\frac{3}{2}\frac{\partial}{\partial t}(P_s) + \nabla\cdot\boldsymbol{Q}_s + q_s\left(\int f_s\boldsymbol{v}\cdot\nabla\phi\mathrm{d}\boldsymbol{v} + \frac{\partial}{\partial t}\int f_s\boldsymbol{v}\cdot\boldsymbol{A}\mathrm{d}\boldsymbol{v}\right) = 0 \tag{31}$$

Now, using the following relation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\phi(\boldsymbol{r},t)f_s(\boldsymbol{r},\boldsymbol{v},t)) &= f_s\left(\frac{\partial\phi}{\partial t} + \nabla\phi\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right) + \phi\left(\frac{\partial f_s}{\partial t} + \nabla f_s\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \frac{\partial f_s}{\partial \boldsymbol{v}}\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right) \\ &= f_s\frac{\partial\phi}{\partial t} + f_s\boldsymbol{v}\cdot\nabla\phi + \phi\left(\frac{\partial f_s}{\partial t}\right)_{\mathrm{coll}},\end{aligned} \tag{32}$$

the ϕ term of the third term of (31) is represented by

$$\begin{aligned}\int f_s\boldsymbol{v}\cdot\nabla\phi\mathrm{d}\boldsymbol{v} &= \int\left(\frac{\mathrm{d}}{\mathrm{d}t}(\phi f_s) - f_s\frac{\partial\phi}{\partial t} - \phi\left(\frac{\partial f_s}{\partial t}\right)_{\mathrm{coll}}\right)\mathrm{d}\boldsymbol{v} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\phi n_s) - \int f_s\frac{\partial\phi}{\partial t}\mathrm{d}\boldsymbol{v}.\end{aligned} \tag{33}$$

Therefore, (31) is written with (33) as

$$\frac{3}{2}\frac{\partial P_s}{\partial t} + \nabla\cdot\boldsymbol{Q}_s = q_s\int f_s\frac{\partial\chi}{\partial t}\mathrm{d}\boldsymbol{v} - \frac{\mathrm{d}}{\mathrm{d}t}(q_s n_s\phi). \tag{34}$$

where $\chi = \phi - \boldsymbol{v}\cdot\boldsymbol{A}$.

3 Derivation of the heating equation

We consider the variation in the equilibrium quantities on the timescale with the order $\mathcal{O}(\epsilon^2\omega)$ slower than the timescale of the variation in the fluctuating quantities $\sim\mathcal{O}(\omega)$. Here, we treat up to the second order fluctuation. (But the component of the second order fluctuation except for the cross term between the first order fluctuations does not contribute to the heating.)

We decompose the physical quantity Ψ into the equilibrium and perturbation parts,

$$\Psi = \Psi_0 + \delta\Psi, \tag{35}$$
$$\delta\Psi = \epsilon\Psi_1 + \epsilon^2\Psi_2 + \mathcal{O}(\epsilon^3). \tag{36}$$

Additionally, the time derivative order is

$$\frac{\partial\Psi_0}{\partial t} \sim \mathcal{O}(\epsilon^2\omega\Psi_0), \tag{37}$$
$$\frac{\partial\delta\Psi}{\partial t} \sim \mathcal{O}(\omega\delta\Psi). \tag{38}$$

We consider the medium timescale t_{m} long compared to the fluctuation timescale $1/\omega$ but short compared to the heating timescale $1/(\epsilon^2\omega)$,

$$\frac{1}{\omega} \ll t_{\mathrm{m}} \ll \frac{1}{\epsilon^2\omega}, \tag{39}$$

and define the medium-time average over a period t_{m},

$$\overline{\Psi}(t) = \frac{1}{t_{\mathrm{m}}}\int_{-t_{\mathrm{m}}/2}^{t_{\mathrm{m}}/2} \Psi(t')dt'. \tag{40}$$

The equilibrium distribution function f_{0s} is constant at times $\sim t_{\mathrm{m}}$, so we write $\overline{f_{0s}} = f_{0s}$.

3.1 Continuity equation

We first consider the continuity equation (23) derived by the Fokker–Planck equation.

$$\epsilon\left(\frac{\partial n_{1s}}{\partial t} + \nabla\cdot(n_{0s}\boldsymbol{u}_{1s})\right) + \epsilon^2\left(\frac{\partial n_{0s}}{\partial t} + \frac{\partial n_{2s}}{\partial t} + \nabla\cdot(n_{1s}\boldsymbol{u}_{1s}) + \nabla\cdot(n_{0s}\boldsymbol{u}_{2s})\right) = 0 \tag{41}$$

Integrating over the real space $\int \mathrm{d}\boldsymbol{r}$,

$$\int \frac{\partial}{\partial t} n_{0s}(x)\mathrm{d}\boldsymbol{r} = 0 \tag{42}$$
$$\Rightarrow \frac{\partial n_{0s}}{\partial t} = 0 \tag{43}$$
$$\Rightarrow n_{0s} = \text{const. in time.} \tag{44}$$

Note that $\int \epsilon\Psi_1\mathrm{d}\boldsymbol{r} = \int \epsilon^2\Psi_2\mathrm{d}\boldsymbol{r} = 0$ and $\int_{-\infty}^{\infty} \nabla\cdot\boldsymbol{\Gamma}\mathrm{d}\boldsymbol{r} = 0$ are used.

3.2 Transport equation

The transport equation derived in (34) is represented by

$$\epsilon\left(\frac{3}{2}\frac{\partial P_{1s}}{\partial t} + \nabla\cdot\boldsymbol{Q}_{1s}\right) + \epsilon^2\left(\frac{3}{2}\frac{\partial P_{0s}}{\partial t} + \frac{3}{2}\frac{\partial P_{2s}}{\partial t} + \nabla\cdot\boldsymbol{Q}_{2s}\right) \tag{45}$$
$$= \epsilon\left(q_s\int f_{0s}\frac{\partial\chi_1}{\partial t}\mathrm{d}\boldsymbol{v} - \frac{d}{dt}(q_s n_{0s}\phi_1)\right) \tag{46}$$
$$+ \epsilon^2\left(q_s\int \delta f_{1s}\frac{\partial\chi_1}{\partial t}\mathrm{d}\boldsymbol{v} + q_s\int f_{0s}\frac{\partial\chi_2}{\partial t}\mathrm{d}\boldsymbol{v} - \frac{\mathrm{d}}{\mathrm{d}t}(q_s n_{1s}\phi_1) - \frac{\mathrm{d}}{\mathrm{d}t}(q_s n_{0s}\phi_2)\right). \tag{47}$$

Integrating over the real space $\int \mathrm{d}\boldsymbol{r}$ and using the assumption $\int \epsilon\Psi_1 \mathrm{d}\boldsymbol{r} = \int \epsilon^2\Psi_2 \mathrm{d}\boldsymbol{r} = 0$ and $\int_{-\infty}^{\infty} \nabla\cdot\boldsymbol{Q}\mathrm{d}\boldsymbol{r} = 0$,

$$\int \frac{3}{2}\frac{\partial P_{0s}}{\partial t}\mathrm{d}\boldsymbol{r} = q_s \int\int \delta f_{1s}\frac{\partial \chi}{\partial t}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{r} - \int \frac{\mathrm{d}}{\mathrm{d}t}(q_s n_{1s}\phi)\mathrm{d}\boldsymbol{r}. \tag{48}$$

Note that the second order but the cross term between the first perturbations remains because this term means the nonlinear interaction so it does not become zero by integrating over the real space. For later discussion, χ_1 and ϕ_1 are denoted by χ and ϕ, respectively.

Using the definition of δf_{1s} corresponding to (3), the relation in the fluid model $P_{0s} = n_{0s}T_{0s}$ and (44), (48) is represented as

$$\frac{1}{V}\int \frac{3}{2}n_{0s}\frac{\partial T_{0s}}{\partial t}\mathrm{d}\boldsymbol{r} = \frac{1}{V}q_s\int\int h_s\frac{\partial\chi}{\partial t}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{r} - \frac{1}{V}q_s\int\int \frac{q_s\phi}{T_{0s}}f_{0s}\frac{\partial\chi}{\partial t}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{r} - \frac{1}{V}\frac{\mathrm{d}}{\mathrm{d}t}\int (q_s n_{1s}\phi)\mathrm{d}\boldsymbol{r} \tag{49}$$

$$= \frac{1}{V}\int\int q_s h_s\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial t}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s - \frac{1}{V}\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{n_{0s}q_s^2\phi^2}{2T_{0s}}\mathrm{d}\boldsymbol{r} - \frac{1}{V}\frac{\mathrm{d}}{\mathrm{d}t}\int (q_s n_{1s}\phi)\mathrm{d}\boldsymbol{r}, \tag{50}$$

where

$$\begin{aligned} -\frac{1}{V}q_s\int\int \frac{q_s\phi}{T_{0s}}f_{0s}\frac{\partial\chi}{\partial t}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{r} &= -\frac{1}{V}\int\int \frac{q_s^2\phi}{T_{0s}}f_{0s}\frac{\partial\phi}{\partial t}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{r} + \frac{1}{V}\int\int \frac{q_s^2\phi}{T_{0s}}f_{0s}\boldsymbol{v}\cdot\frac{\partial\boldsymbol{A}}{\partial t}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{r} \\ &= -\frac{1}{V}\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{n_{0s}q_s^2\phi^2}{2T_{0s}}\mathrm{d}\boldsymbol{r} + \frac{1}{V}\int \frac{q_s^2\phi}{T_{0s}}\boldsymbol{u}_{0s}\cdot\frac{\partial\boldsymbol{A}}{\partial t}\mathrm{d}\boldsymbol{r}, \end{aligned} \tag{51}$$

and $\boldsymbol{u}_{0s} = 0$ are used.

3.3 Heating equation

Coupling (50) with (7) through $\langle\chi\rangle_{\boldsymbol{R}_s}$ term,

$$\begin{aligned} H_s \equiv \frac{1}{V}\int \frac{3}{2}n_{0s}\frac{\partial T_{0s}}{\partial t}\mathrm{d}\boldsymbol{r} &= \frac{1}{V}\int\int \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{T_{0s}}{2f_{0s}}h_s^2\right)\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s - (I_s - D_s) \\ &\quad - \frac{1}{V}\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{n_{0s}q_s^2\phi^2}{2T_{0s}}\mathrm{d}\boldsymbol{r} - \frac{1}{V}\frac{\mathrm{d}}{\mathrm{d}t}\int (q_s n_{1s}\phi)\mathrm{d}\boldsymbol{r} \\ &= \frac{1}{V}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int\int \frac{T_{0s}}{2f_{0s}}\langle h_s^2\rangle_{\boldsymbol{r}}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{r} - \int \frac{n_{0s}q_s^2\phi^2}{2T_{0s}}\mathrm{d}\boldsymbol{r} - \int (q_s n_{1s}\phi)\mathrm{d}\boldsymbol{r}\right] \\ &\quad + D_s - I_s, \end{aligned} \tag{52}$$

where $(1/V)\int 1\mathrm{d}\boldsymbol{r} = 1$. Taking the medium-time average using (40), the second term of (52) vanishes and only the dissipation and injection terms remain as

$$H_s = \overline{D}_s - \overline{I}_s. \tag{53}$$

Memo: Injection term calculation

In the gyrokinetic system, the injection term due to the background temperature and density gradients is given by

$$I_s = -\frac{1}{V}\int\int \frac{T_{0s}}{L_{n_{0s}}}\frac{h_s}{B_0}\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial Y}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s - \frac{1}{V}\int\int \frac{T_{0s}}{L_{T_{0s}}}\frac{h_s}{B_0}\left(\frac{v^2}{v_{\mathrm{th},s}^2}-\frac{3}{2}\right)\frac{\partial\langle\chi\rangle_{\boldsymbol{R}_s}}{\partial Y}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s. \tag{54}$$

h_s and $\langle\chi\rangle_{\boldsymbol{R}_s}$ are the gyrokinetic distribution function and gyrokinetic potential, respectively, and written in the Fourier space as

$$\begin{aligned} h_s &= \sum_{\boldsymbol{k}} h_{s,\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_s}, \\ \langle\chi\rangle_{\boldsymbol{R}_s} &= \sum_{\boldsymbol{k}}\left[J_0(\alpha_s)(\phi_{\boldsymbol{k}} - v_\parallel A_{\parallel,\boldsymbol{k}}) + \frac{T_{0s}}{q_s}\frac{v_\perp^2}{v_{\mathrm{th},s}^2}\frac{2J_1(\alpha_s)}{\alpha_s}\frac{\delta B_{\parallel,\boldsymbol{k}}}{B_0}\right]\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}_s}, \end{aligned} \tag{55}$$

where $J_0(\alpha_s), J_0(\alpha_s)$ are the Bessel function with $\alpha_s = k_\perp v_\perp/\Omega_s$.

Using (55), the injection term (54) is represented by

$$\begin{aligned} I_s =& -\frac{1}{V}\frac{T_{0s}}{L_{n_{0s}}B_0}\int\int\sum_{\boldsymbol{k}'}h_{s,\boldsymbol{k}'}\sum_{\boldsymbol{k}''}(\mathrm{i}k_y'') \\ &\times\left[J_0''(\alpha_s'')(\phi_{\boldsymbol{k}''} - v_\parallel A_{\parallel,\boldsymbol{k}''})\right]\mathrm{e}^{\mathrm{i}(\boldsymbol{k}'+\boldsymbol{k}'')\cdot\boldsymbol{R}_s}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s \\ &-\frac{1}{V}\frac{T_{0s}}{L_{T_{0s}}B_0}\int\int\left(\frac{v^2}{v_{\mathrm{th},s}^2}-\frac{3}{2}\right)\sum_{\boldsymbol{k}'}h_{s,\boldsymbol{k}'}\sum_{\boldsymbol{k}''}(\mathrm{i}k_y'') \\ &\times\left[J_0(\alpha_s)(\phi_{\boldsymbol{k}''} - v_\parallel A_{\parallel,\boldsymbol{k}''}) + \frac{T_{0s}}{q_s}\frac{v_\perp^2}{v_{\mathrm{th},s}^2}\frac{2J_1(\alpha_s)}{\alpha_s''}\frac{\delta B_{\parallel,\boldsymbol{k}''}}{B_0}\right]\mathrm{e}^{\mathrm{i}(\boldsymbol{k}'+\boldsymbol{k}'')\cdot\boldsymbol{R}_s}\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{R}_s. \end{aligned} \tag{56}$$

Only modes satisfying $\boldsymbol{k}'+\boldsymbol{k}''=0$ should be considered because of orthogonality. Thus,

$$\begin{aligned} I_s =& -\frac{T_{0s}}{L_{n_{0s}}B_0}\sum_{\boldsymbol{k}}\Re\left((\mathrm{i}k_y)\int h_{s,\boldsymbol{k}}^*\left[J_0(\alpha_s)(\phi_{\boldsymbol{k}} - v_\parallel A_{\parallel,\boldsymbol{k}}) + \frac{T_{0s}}{q_s}\frac{v_\perp^2}{v_{\mathrm{th},s}^2}\frac{2J_1(\alpha_s)}{\alpha_s}\frac{\delta B_{\parallel,\boldsymbol{k}}}{B_0}\right]\mathrm{d}\boldsymbol{v}\right) \\ &-\frac{T_{0s}}{L_{T_{0s}}B_0}\sum_{\boldsymbol{k}}\Re\left((\mathrm{i}k_y)\int\left(\frac{v^2}{v_{\mathrm{th},s}^2}-\frac{3}{2}\right)h_{s,\boldsymbol{k}}^*\left[J_0(\alpha_s)(\phi_{\boldsymbol{k}} - v_\parallel A_{\parallel,\boldsymbol{k}}) + \frac{T_{0s}}{q_s}\frac{v_\perp^2}{v_{\mathrm{th},s}^2}\frac{2J_1(\alpha_s)}{\alpha_s}\frac{\delta B_{\parallel,\boldsymbol{k}}}{B_0}\right]\mathrm{d}\boldsymbol{v}\right) \\ =& -\frac{T_{0s}}{B_0}\left(\frac{1}{L_{n_{0s}}}-\frac{3}{2L_{T_{0s}}}\right)\sum_{\boldsymbol{k}}\Re\left((\mathrm{i}k_y)\int h_{s,\boldsymbol{k}}^*\left[J_0(\alpha_s)(\phi_{\boldsymbol{k}} - v_\parallel A_{\parallel,\boldsymbol{k}}) + \frac{T_{0s}}{q_s}\frac{v_\perp^2}{v_{\mathrm{th},s}^2}\frac{2J_1(\alpha_s)}{\alpha_s}\frac{\delta B_{\parallel,\boldsymbol{k}}}{B_0}\right]\mathrm{d}\boldsymbol{v}\right) \\ &-\frac{T_{0s}}{L_{T_{0s}}B_0}\sum_{\boldsymbol{k}}\Re\left((\mathrm{i}k_y)\int\frac{v^2}{v_{\mathrm{th},s}^2}h_{s,\boldsymbol{k}}^*\left[J_0(\alpha_s)(\phi_{\boldsymbol{k}} - v_\parallel A_{\parallel,\boldsymbol{k}}) + \frac{T_{0s}}{q_s}\frac{v_\perp^2}{v_{\mathrm{th},s}^2}\frac{2J_1(\alpha_s)}{\alpha_s}\frac{\delta B_{\parallel,\boldsymbol{k}}}{B_0}\right]\mathrm{d}\boldsymbol{v}\right), \end{aligned} \tag{57}$$

The first and second terms in (57) correspond to the particle and heat fluxes, respectively. Now, we define these flux terms as

$$\begin{aligned} I_{s,\mathrm{pf}} \equiv& -\frac{T_{0s}}{B_0}\left(\frac{1}{L_{n_{0s}}}-\frac{3}{2L_{T_{0s}}}\right)\sum_{\boldsymbol{k}}\Re\left((\mathrm{i}k_y)\int h_{s,\boldsymbol{k}}^*\left[J_0(\alpha_s)(\phi_{\boldsymbol{k}} - v_\parallel A_{\parallel,\boldsymbol{k}}) + \frac{T_{0s}}{q_s}\frac{v_\perp^2}{v_{\mathrm{th},s}^2}\frac{2J_1(\alpha_s)}{\alpha_s}\frac{\delta B_{\parallel,\boldsymbol{k}}}{B_0}\right]\mathrm{d}\boldsymbol{v}\right) \\ =& -\frac{T_{0s}}{B_0}\left(\frac{1}{L_{n_{0s}}}-\frac{3}{2L_{T_{0s}}}\right)\times \\ &\sum_{\boldsymbol{k}}\Re\left((\mathrm{i}k_y)\left[\phi_{\boldsymbol{k}}\int J_0(\alpha_s)h_{s,\boldsymbol{k}}^*\mathrm{d}\boldsymbol{v} - A_{\parallel,\boldsymbol{k}}\int J_0(\alpha_s)v_\parallel h_{s,\boldsymbol{k}}^*\mathrm{d}\boldsymbol{v} + \frac{\delta B_{\parallel,\boldsymbol{k}}}{B_0}\int\frac{T_{0s}}{q_s}\frac{v_\perp^2}{v_{\mathrm{th},s}^2}\frac{2J_1(\alpha_s)}{\alpha_s}h_{s,\boldsymbol{k}}^*\mathrm{d}\boldsymbol{v}\right]\right) \end{aligned} \tag{58}$$

$$I_{s,\mathrm{hf}} \equiv -\frac{T_{0s}}{L_{T_{0s}}B_0}\sum_{\boldsymbol{k}}\Re\left((\mathrm{i}k_y)\int\int\frac{v^2}{v_{\mathrm{th},s}^2}h_{s,\boldsymbol{k}}^*\left[J_0(\alpha_s)(\phi_{\boldsymbol{k}} - v_\parallel A_{\parallel,\boldsymbol{k}}) + \frac{T_{0s}}{q_s}\frac{v_\perp^2}{v_{\mathrm{th},s}^2}\frac{2J_1(\alpha_s)}{\alpha_s}\frac{\delta B_{\parallel,\boldsymbol{k}}}{B_0}\right]\mathrm{d}\boldsymbol{v}\right)$$

$$= -\frac{T_{0s}}{L_{T_{0s}}B_0}\times$$

$$\sum_{\boldsymbol{k}}\Re\left((\mathrm{i}k_y)\left[\phi_{\boldsymbol{k}}\int J_0(\alpha_s)\frac{v^2}{v_{\mathrm{th},s}^2}h_{s,\boldsymbol{k}}^*\mathrm{d}\boldsymbol{v} - A_{\parallel,\boldsymbol{k}}\int J_0(\alpha_s)v_\parallel\frac{v^2}{v_{\mathrm{th},s}^2}h_{s,\boldsymbol{k}}^*\mathrm{d}\boldsymbol{v} + \frac{\delta B_{\parallel,\boldsymbol{k}}}{B_0}\int\frac{T_{0s}}{q_s}\frac{v_\perp^2}{v_{\mathrm{th},s}^2}\frac{2J_1(\alpha_s)}{\alpha_s}\frac{v^2}{v_{\mathrm{th},s}^2}h_{s,\boldsymbol{k}}^*\mathrm{d}\boldsymbol{v}\right]\right) \tag{59}$$

The injection term is represented by $I_s = I_{s,\mathrm{hf}} + I_{s,\mathrm{pf}}$.