Quasi–linear analysis of microtearing turbulence

Mitsuyoshi Yagyu

September 1, 2026

1 Gyrokinetic equation

A two dimensional gyrokinetic equation having the magnetic shear is described as

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{B_y^{\mathrm{eq}}(x)}{B_0}\frac{\partial h_s}{\partial Y_s} + \frac{1}{B_0}\{\langle\chi\rangle_{\boldsymbol{R}_s}, h_s\} = \frac{q_s f_{0s}}{T_{0s}}\left(\frac{\partial}{\partial t} + v_{*,s}^T(v)\frac{\partial}{\partial Y_s}\right)\langle\chi\rangle_{\boldsymbol{R}_s} + C(h_s). \tag{1}$$

The magnetic curvature and drift are ignored. Some definitions are given below:

$$\langle\chi\rangle_{\boldsymbol{R}_s} = \langle\phi - \boldsymbol{v}\cdot\boldsymbol{A}\rangle_{\boldsymbol{R}_s} = \sum_{\boldsymbol{k}} J_0(\alpha_s)(\phi_{\boldsymbol{k}} - v_\parallel A_{\parallel,\boldsymbol{k}}) + \frac{T_{0s}}{q_s}\frac{2v_\perp^2}{v_{\mathrm{th},s}^2}\frac{J_1(\alpha_s)}{\alpha_s}\frac{\delta B_{\parallel,\boldsymbol{k}}}{B_0}, \tag{2}$$

$$v_{*,s}(v)^T \equiv -\frac{T_{0s}}{q_s B_0}\left(\frac{1}{L_{n_{0s}}} + \left[\frac{v^2}{v_{\mathrm{th},s}^2} - \frac{3}{2}\right]\frac{1}{L_{T_{0s}}}\right) = v_{*,n_{0s}}\left(1 + \left[\frac{v^2}{v_{\mathrm{th},s}^2} - \frac{3}{2}\right]\eta_s\right), \tag{3}$$

$$f_{0s} = \frac{n_{0s}}{(\sqrt{\pi}v_{\mathrm{th},s})^3}\exp(-v_\parallel^2/v_{\mathrm{th},s}^2 - v_\perp^2/v_{\mathrm{th},s}^2) \tag{4}$$

where J_0, J_1 are zeroth- and first-order Bessel functions with $\alpha_s = k_\perp v_\perp/\Omega_s$, respectively. $s = \mathrm{e}, \mathrm{i}$ is species type.

2 Review of Drake+, 1980

2.1 Basic equation

Consider the drift-kinetic equation without the magnetic shear for electrons. Furthermore, the electrostatic potential ϕ and the magnetic compressibility $\delta B_\parallel$ are ignored here. Then, (1) is represented by

$$\frac{\partial h_{\mathrm{e}}}{\partial t} - \frac{v_\parallel}{B_0}\{A_\parallel, h_{\mathrm{e}}\} = -v_\parallel\frac{q_{\mathrm{e}} f_{0\mathrm{e}}}{T_{0\mathrm{e}}}\left(\frac{\partial}{\partial t} + v_{*,\mathrm{e}}^T(v)\frac{\partial}{\partial y}\right)A_\parallel + C(h_{\mathrm{e}}). \tag{5}$$

The Lorentz collision operator is employed for the collision term, which is defined as

$$C(h_{\mathrm{e}}) = \frac{\nu_{\mathrm{ei}}(v)}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}h_{\mathrm{e}}, \tag{6}$$

where $\xi \equiv v_\parallel/v$ is the pitch angle variable.

Using the relation

$$\frac{v_\parallel}{B_0}\{A_\parallel, h_{\mathrm{e}}\} = -v_\parallel\frac{\boldsymbol{B}_\perp}{B_0}\cdot\nabla_\perp h_{\mathrm{e}}, \tag{7}$$

the equation corresponding to Eq.(6) in [Drake+, 1980] is obtained as

$$\left(\frac{\partial}{\partial t} + v_\parallel\frac{\boldsymbol{B}_\perp}{B_0}\cdot\nabla_\perp - \frac{\nu_{\mathrm{ei}}(v)}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}\right)h_{\mathrm{e}} = -v_\parallel\frac{q_{\mathrm{e}} f_{0\mathrm{e}}}{T_{0\mathrm{e}}}\left(\frac{\partial}{\partial t} + v_{*,\mathrm{e}}^T(v)\frac{\partial}{\partial y}\right)A_\parallel. \tag{8}$$

2.1.1 Linear solution

As a first step, we obtain the linear solution for the microtearing mode without the magnetic shear. Consider the small perturbation but larger than the gyrokinetic ordering parameter, $\mathcal{O}(\epsilon) \ll \mathcal{O}(\delta) \ll 1$, namely, $\boldsymbol{B}_\perp \sim A_\parallel \sim h_\mathrm{e} \sim \mathcal{O}(\delta)$. (8) is linearized by neglecting the nonlinear term with the second-order in δ, yielding

$$\delta^1: \quad \left(\frac{\partial}{\partial t} - \frac{\nu_\mathrm{ei}(v)}{2}\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi}\right) h_\mathrm{e} = -v_\parallel \frac{q_\mathrm{e} f_{0\mathrm{e}}}{T_{0\mathrm{e}}}\left(\frac{\partial}{\partial t} + v^T_{*,\mathrm{e}}(v)\frac{\partial}{\partial y}\right) A_\parallel. \tag{9}$$

In the following, the subscripts 'e' and 'ei' are omitted.

Decompose h and $A_\parallel$ into Fourier modes(Use k_y instead of $\boldsymbol{k}$ in linear analysis. Rewrite it!),

$$h = \sum_{\boldsymbol{k}} h_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}} t)}, \tag{10}$$

$$A_\parallel = \sum_{\boldsymbol{k}} A_{\parallel,\boldsymbol{k}} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}} t)}, \tag{11}$$

to obtain

$$\left(-\mathrm{i}\omega_{\boldsymbol{k}} - \frac{\nu(v)}{2}\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi}\right) h_{\boldsymbol{k}} = \mathrm{i} v_\parallel \frac{q f_0}{T_0}\left(\omega_{\boldsymbol{k}} - \omega^T_{*,k_y}(v)\right) A_{\parallel,\boldsymbol{k}}, \tag{12}$$

where $\omega^T_{*,k_y}(v) \equiv k_y v^T_*(v)$.

Expanding $h_{\boldsymbol{k}}$ by Legendre polynomials $h_{\boldsymbol{k}} = \sum_n h_n(v) P_n(\xi)$,

$$\sum_n \left(-\mathrm{i}\omega_{\boldsymbol{k}} + \frac{\nu(v)}{2} n(n+1)\right) h_n(v) P_n(\xi) = \mathrm{i} v \xi \frac{q f_0}{T_0}\left(\omega_{\boldsymbol{k}} - \omega^T_{*,k_y}(v)\right) A_{\parallel,\boldsymbol{k}}, \tag{13}$$

and multiplying (13) by $P_m(\xi)$, the orthogonality of the Legendre polynomials yields

$$\text{LHS of (13)} = \sum_n \left(-\mathrm{i}\omega_{\boldsymbol{k}} + \frac{\nu(v)}{2} n(n+1)\right) h_n(v) \int_{-1}^{1} P_n(\xi) P_m(\xi) \mathrm{d}\xi = [\cdots] h_m(v) \frac{2}{2m+1}, \tag{14}$$

$$\text{RHS of (13)} = \mathrm{i} v \frac{q f_0}{T_0}\left(\omega_{\boldsymbol{k}} - \omega^T_{*,k_y}(v)\right) A_{\parallel,\boldsymbol{k}} \int_{-1}^{1} \xi P_m(\xi) \mathrm{d}\xi, \rightarrow m = 1, \tag{15}$$

then,

$$h_1 = \mathrm{i} v \frac{q f_0}{T_0} \frac{\omega_{\boldsymbol{k}} - \omega^T_{*,k_y}(v)}{\nu(v) - \mathrm{i}\omega_{\boldsymbol{k}}} A_{\parallel,\boldsymbol{k}} = \mathrm{i} v \frac{q f_0}{T_0} \frac{\omega_{\boldsymbol{k}} - \omega_{*,n_0} - \omega_{*,T_0}(\hat{v}^2 - 3/2)}{\nu(v) - \mathrm{i}\omega_{\boldsymbol{k}}} A_{\parallel,\boldsymbol{k}}. \tag{16}$$

where $\hat{v} = v/v_\mathrm{th}$, $\omega_{*,n_0,T_0} \equiv k_y v_{*,n_0,T_0}$.Therefore, $h_{\mathrm{e},\boldsymbol{k}} = P_1(\xi) h_1(v)$.

Using Ampère's law to obtain the electric conductivity σ,

$$
\begin{aligned}
j_{\parallel,\boldsymbol{k}} &= \sum_s q_s \int \langle v_\parallel h_{s,\boldsymbol{k}} \rangle_{\boldsymbol{r}} \mathrm{d}\boldsymbol{v} && (17)\\
&\approx q_\mathrm{e} \int v_\parallel h_{\mathrm{e},\boldsymbol{k}} \mathrm{d}\boldsymbol{v} && (18)\\
&= q \int v \xi^2 h_1(v) 2\pi v^2 \mathrm{d}v \mathrm{d}\xi && (19)\\
&= 2\pi q \int_0^\infty v^3 h_1(v) \mathrm{d}v \int_{-1}^{1} \xi^2 \mathrm{d}\xi && (20)\\
&= \frac{4\pi}{3} q \int_0^\infty v^3 h_1(v) \mathrm{d}v && (21)\\
&= \frac{4\pi}{3} \frac{q^2}{T_0} \int_0^\infty \frac{v^4 f_0(v)}{\nu(v)} \frac{1 - \omega_{*,n_0}/\omega_{\boldsymbol{k}} - \omega_{*,T_0}/\omega_{\boldsymbol{k}}(\hat{v}^2 - 3/2)}{1 - \mathrm{i}\omega_{\boldsymbol{k}}/\nu(v)} \mathrm{d}v (\mathrm{i}\omega_{\boldsymbol{k}} A_{\parallel,\boldsymbol{k}}) && (22)\\
&\approx \underbrace{\frac{4\pi}{3} \frac{q^2}{T_0} \int_0^\infty \frac{v^4 f_0(v)}{\nu(v)} \left(1 - \frac{\omega_{*,n_0}}{\omega_{\boldsymbol{k}}} - \frac{\omega_{*,T_0}}{\omega_{\boldsymbol{k}}} \left[\hat{v}^2 - \frac{3}{2}\right]\right) \left(1 + \frac{\mathrm{i}\omega_{\boldsymbol{k}}}{\nu(v)}\right) \mathrm{d}v}_{\sigma} \, E_{\parallel,\boldsymbol{k}} && (23)\\
&= \sigma(\omega, k) E_{\parallel,\boldsymbol{k}}, && (24)
\end{aligned}
$$

where the parallel flow for ions is ignored due to $u_{\parallel,\mathrm{e}} \gg u_{\parallel,\mathrm{i}}$. The subsidiary ordering $\omega/\nu \ll 1$ is used. (Transforming the velocity-coordinates from $(v_\parallel, v_\perp)$ to (v, ξ), Jacobian yields the volume element $\mathrm{d}\boldsymbol{v} = \mathrm{d}v v^2 \mathrm{d}\xi \mathrm{d}\theta$.)

Here, $\nu(v)$ [H&S eq.(3.31)] and $f_0(v)$ are, respectively, given by

$$
\nu(v) = \frac{3}{4}\sqrt{\pi}\nu_\mathrm{c}\hat{v}^{-3}, \quad f_0(v) = \frac{n_0}{(\sqrt{\pi} v_\mathrm{th})^3} \mathrm{e}^{-\hat{v}^2}. \tag{25}
$$

Then, σ is calculated as

$$
\begin{aligned}
\sigma &= \frac{4\pi}{3} \frac{q^2}{T_0} \int_0^\infty \frac{v^4 f_0(v)}{\nu(v)} \left(1 - \frac{\omega_{*,n_0}}{\omega_{\boldsymbol{k}}} - \frac{\omega_{*,T_0}}{\omega_{\boldsymbol{k}}} \left[\hat{v}^2 - \frac{3}{2}\right]\right) \left(1 + \frac{\mathrm{i}\omega_{\boldsymbol{k}}}{\nu(v)}\right) \mathrm{d}v && (26)\\
&= \frac{8\pi}{3\sqrt{\pi}} \frac{\omega_\mathrm{p}^2}{\mu_0 \mathrm{c}^2} \int_0^\infty \frac{\hat{v}^4 f_0(v)}{\nu(v)} \left(1 - \frac{\omega_{*,n_0}}{\omega_{\boldsymbol{k}}} - \frac{\omega_{*,T_0}}{\omega_{\boldsymbol{k}}} \left[\hat{v}^2 - \frac{3}{2}\right]\right) \left(1 + \frac{\mathrm{i}\omega_{\boldsymbol{k}}}{\nu(v)}\right) \mathrm{d}\hat{v} && (27)\\
&= \text{...omit detailed calc.} && (28)\\
&= \frac{32}{3\pi\nu_\mathrm{c}} \frac{\omega_\mathrm{p}^2}{\mu_0 \mathrm{c}^2} \left[1 - \frac{\omega_{*,n_0}}{\omega_{\boldsymbol{k}}} \left(1 + \frac{5}{2}\eta_\mathrm{e}\right) + \frac{105}{16} \frac{\mathrm{i}\omega_{\boldsymbol{k}}}{\nu_\mathrm{c}} \left(1 - \frac{\omega_{*,n_0}}{\omega_{\boldsymbol{k}}} (1 + 4\eta_\mathrm{e})\right)\right] && (29)
\end{aligned}
$$

Substituting (29) into (24),

$$
j_{\parallel,\boldsymbol{k}} = \frac{k_\perp^2 A_{\parallel,\boldsymbol{k}}}{\mu_0} = \frac{32}{3\pi\nu_\mathrm{c}} \frac{\omega_\mathrm{p}^2}{\mu_0 \mathrm{c}^2} \left[1 - \frac{\omega_{*,n_0}}{\omega_{\boldsymbol{k}}} \left(1 + \frac{5}{2}\eta_\mathrm{e}\right) + \frac{105}{16} \frac{\mathrm{i}\omega_{\boldsymbol{k}}}{\nu_\mathrm{c}} \left(1 - \frac{\omega_{*,n_0}}{\omega_{\boldsymbol{k}}} (1 + 4\eta_\mathrm{e})\right)\right] \times \mathrm{i}\omega_{\boldsymbol{k}} A_{\parallel,\boldsymbol{k}}. \tag{30}
$$

Therefore, the dispersion relation is obtained as

$$
\alpha_\eta (k_\perp d)^2 = \mathrm{i} \left[\frac{\omega_{\boldsymbol{k}}}{\nu_\mathrm{c}} - \frac{\omega_*}{\nu_\mathrm{c}} + \frac{105}{16} \frac{\mathrm{i}\omega_{\boldsymbol{k}}}{\nu_\mathrm{c}} \left(\frac{\omega_{\boldsymbol{k}}}{\nu_\mathrm{c}} - \frac{\omega_{*'}}{\nu_\mathrm{c}}\right)\right] \tag{31}
$$

where $\alpha_\eta \equiv 3\pi/32$, $d^2 \equiv \mathrm{c}^2/\omega_\mathrm{p}^2$, $\omega_* \equiv \omega_{*,n_0}(1 + 5\eta_\mathrm{e}/2)$ and $\omega_{*'} \equiv \omega_{*,n_0}(1 + 4\eta_\mathrm{e})$.

Looking for a solution of the order $\omega/\nu \sim \omega_*/\nu \sim k_\perp d \sim \mathcal{O}(\delta) \ll 1$ due to the perturbative expansion of $\omega = \omega_0 + \delta\omega_1$,

$$
\begin{aligned}
\omega_0 &= \omega_*, && (32)\\
\omega_1 &= \mathrm{i}\left(\frac{3}{2} \frac{105}{16} \frac{\omega_*}{\nu_\mathrm{c}} \omega_{*,T_0} - \alpha_\eta \nu_\mathrm{c} (k_\perp d)^2\right). && (33)
\end{aligned}
$$

Thus, the linear solution $\omega_{\boldsymbol{k}}^{\mathrm{lin}}$ is given by

$$\omega_{\boldsymbol{k}}^{\mathrm{lin}} = \omega_* + \mathrm{i}\left(\alpha\alpha'\frac{\omega_*}{\nu_{\mathrm{ei}}}\omega_{*,T_0} - \alpha_\eta\nu_{\mathrm{ei}}(k_\perp d)^2\right), \tag{34}$$

where $\alpha = 3/2, \alpha' = 105/16$. ν_{c} is replaced by ν_{ei}.

2.1.2 Iteratively perturbative analysis

The nonlinear term in (5) using Bonnet's recursion for the Legendre polynomials is written as

$$v_\parallel\frac{\boldsymbol{B}_\perp}{B_0}\cdot\nabla_\perp h = \sum_n \xi v\frac{\boldsymbol{B}_\perp}{B_0}\cdot\nabla_\perp h_n(v)P_n(\xi) \tag{35}$$

$$= v\frac{\boldsymbol{B}_\perp}{B_0}\cdot\nabla_\perp h_n(v)\frac{(n+1)P_{n+1}+nP_{n-1}}{2n+1}. \tag{36}$$

Multiplying (5) by $P_m(\xi)$ and taking the orthogonality for the Legendre polynomials,

$$\sum_n\frac{\partial h_n(v)}{\partial t}\int_{-1}^{1}P_n(\xi)P_m(\xi)\mathrm{d}\xi = \sum_n\frac{\partial h_n(v)}{\partial t}\frac{2}{2n+1}\delta_{m,n}, \tag{37}$$

$$\begin{aligned} &v\frac{\boldsymbol{B}_\perp}{B_0}\cdot\nabla_\perp\sum_n h_n(v)\left(\int_{-1}^{1}\frac{n+1}{2n+1}P_{n+1}(\xi)P_m(\xi)\mathrm{d}\xi + \int_{-1}^{1}\frac{n}{2n+1}P_{n-1}(\xi)P_m(\xi)\mathrm{d}\xi\right)\\ &= v\frac{\boldsymbol{B}_\perp}{B_0}\cdot\nabla_\perp\sum_n h_n(v)\left(\frac{2(n+1)}{(2n+1)(2n+3)}\delta_{m,n+1} + \frac{2n}{(2n+1)(2n-1)}\delta_{m,n-1}\right), \end{aligned} \tag{38}$$

$$-v\frac{qf_0}{T_0}\left(\frac{\partial}{\partial t} + v_*^T(v)\frac{\partial}{\partial t}\right)A_\parallel\int_{-1}^{1}P_1(\xi)P_m\mathrm{d}\xi = -\frac{2v}{3}\frac{qf_0}{T_0}\left(\frac{\partial}{\partial t} + v_*^T(v)\frac{\partial}{\partial t}\right)A_\parallel\delta_{m,1}, \tag{39}$$

$$-\frac{\nu(v)}{2}\sum_n n(n+1)h_n(v)\int_{-1}^{1}P_nP_m\mathrm{d}\xi = -\frac{\nu(v)}{2}\sum_n n(n+1)h_n(v)\frac{2}{2n+1}\delta_{m,n}. \tag{40}$$

The drift-kinetic equation for electrons using Legendre polynomials is eventually described as

$$\begin{aligned} &\sum_n\left(\frac{\partial}{\partial t} + \frac{\nu(v)}{2}n(n+1)\right)h_n(v)\frac{2}{2n+1}\delta_{m,n}\\ &\quad + v\frac{\boldsymbol{B}_\perp}{B_0}\cdot\nabla_\perp\sum_n h_n(v)\left(\frac{2(n+1)}{(2n+1)(2n+3)}\delta_{m,n+1} + \frac{2n}{(2n+1)(2n-1)}\delta_{m,n-1}\right)\\ &\quad = -\frac{2v}{3}\frac{qf_0}{T_0}\left(\frac{\partial}{\partial t} + v_*^T(v)\frac{\partial}{\partial y}\right)A_\parallel\delta_{m,1}. \end{aligned} \tag{41}$$

For each m in (41), the equations are organized and expressed in terms of a Fourier series expansion $\psi(\boldsymbol{r},t) = \sum_{\boldsymbol{k}}\psi_{\boldsymbol{k}}\exp\left[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}}t)\right]$, where $\omega(\boldsymbol{k}) \equiv \omega_{\boldsymbol{k}}$. *

---

*The three-wave resonant condition

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}, \tag{I}$$

$$\omega_{\boldsymbol{k}_1} + \omega_{\boldsymbol{k}_2} = \omega_{\boldsymbol{k}}, \tag{II}$$

is assumed here.

(i) $m = 0$

$$\frac{\partial}{\partial t} h_0 = -\frac{v}{3} \frac{\boldsymbol{B}_\perp}{B_0} \cdot \nabla_\perp h_1 \tag{42}$$

$$\Rightarrow h_{0,\boldsymbol{k}} = \frac{v}{3\omega_{\boldsymbol{k}}} \sum_{\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}} \frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}}{B_0} \cdot \boldsymbol{k}_2 h_{1,\boldsymbol{k}_2} \tag{43}$$

(ii) $m = 1$

$$\left(\frac{\partial}{\partial t} + \nu(v)\right) h_1 + \frac{\boldsymbol{B}_\perp}{B_0} \cdot \nabla_\perp \left(h_0 + \frac{2}{5} h_2\right) = -v \frac{q f_0}{T_0} \left(\frac{\partial}{\partial t} + v_*^T(v) \frac{\partial}{\partial y}\right) A_\parallel \tag{44}$$

$$\Rightarrow h_{1,\boldsymbol{k}} = -\frac{\mathrm{i}v}{\nu(v) - \mathrm{i}\omega_{\boldsymbol{k}}} \sum_{\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}} \frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}}{B_0} \cdot \boldsymbol{k}_2 \left(h_{0,\boldsymbol{k}_2} + \frac{2}{5} h_{2,\boldsymbol{k}_2}\right) + \frac{\mathrm{i}v}{\nu(v) - \mathrm{i}\omega_{\boldsymbol{k}}} \frac{q f_0}{T_0} \left(\omega_{\boldsymbol{k}} - \omega_{*,\boldsymbol{k}}^T(v)\right) A_{\parallel,\boldsymbol{k}}, \tag{45}$$

where $\omega_{*,\boldsymbol{k}}^T(v) \equiv k_y v_*^T(v)$.

(iii) $m = 2$ Higher–order components ($m \geq 3$) are negligible due to sufficiently strong particle collisions.

$$\left(\frac{\partial}{\partial t} + 3\nu(v)\right) h_2 = -v \frac{\boldsymbol{B}_\perp}{B_0} \cdot \nabla_\perp \left(\frac{2}{3} h_1 + \frac{3}{7} h_3\right) \approx -v \frac{\boldsymbol{B}_\perp}{B_0} \cdot \nabla_\perp \frac{2}{3} h_1 \tag{46}$$

$$\Rightarrow h_{2,\boldsymbol{k}} = -\frac{2\mathrm{i}v}{3(3\nu(v) - \mathrm{i}\omega_{\boldsymbol{k}})} \sum_{\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}} \frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}}{B_0} \cdot \boldsymbol{k}_2 h_{1,\boldsymbol{k}_2} \tag{47}$$

We suppose that a small perturbation, $h \sim B \sim A_\parallel \sim \epsilon \ll 1$, is imposed on (43)- (47), and we organize the equations order by order in ϵ.

ϵ^1:

$$h_{1,\boldsymbol{k}}^{(1)} = \frac{\mathrm{i}v}{\nu(v) - \mathrm{i}\omega_{\boldsymbol{k}}} \frac{q f_0}{T_0} \left(\omega_{\boldsymbol{k}} - \omega_{*,\boldsymbol{k}}^T(v)\right) A_{\parallel,\boldsymbol{k}}^{(1)}. \tag{48}$$

ϵ^2:

$$h_{0,\boldsymbol{k}}^{(2)} = \frac{v}{3\omega_{\boldsymbol{k}}} \sum_{\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}} \frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}^{(1)}}{B_0} \cdot \boldsymbol{k}_2 h_{1,\boldsymbol{k}_2}^{(1)}, \tag{49}$$

$$h_{2,\boldsymbol{k}}^{(2)} = -\frac{2\mathrm{i}v}{3(3\nu(v) - \mathrm{i}\omega_{\boldsymbol{k}})} \sum_{\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}} \frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}^{(1)}}{B_0} \cdot \boldsymbol{k}_2 h_{1,\boldsymbol{k}_2}^{(1)}. \tag{50}$$

ϵ^3:

$$h_{1,\boldsymbol{k}}^{(3)} = -\frac{\mathrm{i}v}{\nu(v) - \mathrm{i}\omega_{\boldsymbol{k}}} \sum_{\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}} \frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}^{(1)}}{B_0} \cdot \boldsymbol{k}_2 \left(h_{0,\boldsymbol{k}_2}^{(2)} + \frac{2}{5} h_{2,\boldsymbol{k}_2}^{(2)}\right). \tag{51}$$

Substituting (48) into (49) and (50),

$$h_{0,\boldsymbol{k}}^{(2)} = \frac{v}{3\omega_{\boldsymbol{k}}} \sum_{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}} \left(\frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}^{(1)}}{B_0} \cdot \boldsymbol{k}_2 \right) \frac{\mathrm{i}v}{\nu(v)-\mathrm{i}\omega_{\boldsymbol{k}_2}} \frac{qf_0}{T_0} \left(\omega_{\boldsymbol{k}_2} - \omega_{*,\boldsymbol{k}_2}^T(v)\right) A_{\parallel,\boldsymbol{k}_2}^{(1)}$$

$$= \frac{\mathrm{i}v^2}{3\omega_{\boldsymbol{k}}} \frac{qf_0}{T_0} \sum_{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}} \left(\frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}^{(1)}}{B_0} \cdot \boldsymbol{k}_2 \right) \frac{\omega_{\boldsymbol{k}_2} - \omega_{*,\boldsymbol{k}_2}^T(v)}{\nu(v)-\mathrm{i}\omega_{\boldsymbol{k}_2}} A_{\parallel,\boldsymbol{k}_2}^{(1)}, \tag{52}$$

$$h_{2,\boldsymbol{k}}^{(2)} = -\mathrm{i}\frac{2v}{3(3\nu(v))-\mathrm{i}\omega_{\boldsymbol{k}})} \sum_{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}} \left(\frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}^{(1)}}{B_0} \cdot \boldsymbol{k}_2 \right) \frac{\mathrm{i}v}{\nu(v)-\mathrm{i}\omega_{\boldsymbol{k}_2}} \frac{qf_0}{T_0} \left(\omega_{\boldsymbol{k}_2} - \omega_{*,\boldsymbol{k}_2}^T(v)\right) A_{\parallel,\boldsymbol{k}_2}^{(1)}$$

$$= \frac{2v^2}{3(3\nu(v)-\mathrm{i}\omega_{\boldsymbol{k}})} \frac{qf_0}{T_0} \sum_{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}} \left(\frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}^{(1)}}{B_0} \cdot \boldsymbol{k}_2 \right) \frac{\omega_{\boldsymbol{k}_2} - \omega_{*,\boldsymbol{k}_2}^T(v)}{\nu(v)-\mathrm{i}\omega_{\boldsymbol{k}_2}} A_{\parallel,\boldsymbol{k}_2}^{(1)} \tag{53}$$

and $h_{0,\boldsymbol{k}}^{(2)}, h_{2,\boldsymbol{k}}^{(2)}$ are substituted into (51) to obtain

$$h_{1,\boldsymbol{k}}^{(3)} = -\frac{\mathrm{i}v}{\nu(v)-\mathrm{i}\omega_{\boldsymbol{k}}} \sum_{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}} \left(\frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}^{(1)}}{B_0} \cdot \boldsymbol{k}_2 \right) \frac{\mathrm{i}v^2}{3} \frac{qf_0}{T_0} \left(\frac{1}{\omega_{\boldsymbol{k}_2}} - \mathrm{i}\frac{4}{5}\frac{1}{3\nu(v)-\mathrm{i}\omega_{\boldsymbol{k}_2}} \right)$$
$$\times \sum_{\boldsymbol{k}_2+\boldsymbol{k}_4=\boldsymbol{k}_2} \left(\frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_3}^{(1)}}{B_0} \cdot \boldsymbol{k}_4 \right) \frac{\omega_{\boldsymbol{k}_4} - \omega_{*,\boldsymbol{k}_4}^T(v)}{\nu(v)-\mathrm{i}\omega_{\boldsymbol{k}_4}} A_{\parallel,\boldsymbol{k}_4}^{(1)} \tag{54}$$

$$= \frac{v^3}{\nu(v)-\mathrm{i}\omega_{\boldsymbol{k}}} \frac{qf_0}{3T_0} \sum_{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}} \sum_{\boldsymbol{k}_2+\boldsymbol{k}_4=\boldsymbol{k}_2} \left(\frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}^{(1)}}{B_0} \cdot \boldsymbol{k}_2 \right) \left(\frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_3}^{(1)}}{B_0} \cdot \boldsymbol{k}_4 \right) A_{\parallel,\boldsymbol{k}_4}^{(1)}$$
$$\left(\frac{1}{\omega_{\boldsymbol{k}_2}} - \mathrm{i}\frac{4}{5}\frac{1}{3\nu(v)-\mathrm{i}\omega_{\boldsymbol{k}_2}} \right) \frac{\omega_{\boldsymbol{k}_4} - \omega_{*,\boldsymbol{k}_4}^T(v)}{\nu(v)-\mathrm{i}\omega_{\boldsymbol{k}_4}} \tag{55}$$

$h_{\boldsymbol{k}}$ is eventually expanded as

$$h_{\boldsymbol{k}}(v,\xi) = \sum_n h_{n,\boldsymbol{k}}(v) P_n(\xi) = \underbrace{h_{1,\boldsymbol{k}}^{(1)}(v)P_1(\xi)}_{\mathcal{O}(\epsilon)} + \underbrace{\left(h_{0,\boldsymbol{k}}^{(2)}(v)P_0(\xi) + h_{2,\boldsymbol{k}}^{(2)}(v)P_2(\xi)\right)}_{\mathcal{O}(\epsilon^2)} + \underbrace{h_{1,\boldsymbol{k}}^{(3)}(v)P_1(\xi)}_{\mathcal{O}(\epsilon^3)}. \tag{56}$$

Taking the $v_\parallel$ moment of (56), we obtain

$$\begin{aligned} j_{\parallel,\boldsymbol{k}} &= q\int v_\parallel h_{\boldsymbol{k}}(v,\xi)\mathrm{d}\boldsymbol{v} \\ &= q\int v\xi h_{\boldsymbol{k}}(v,\xi)\mathrm{d}v v^2 \mathrm{d}\xi \mathrm{d}\theta \\ &= q\int 2\pi v^3 \xi^2 \left(h_{1,\boldsymbol{k}}^{(1)} + h_{1,\boldsymbol{k}}^{(3)}\right) \mathrm{d}v\mathrm{d}\xi, \end{aligned} \tag{57}$$

where the contributions from $h_{0,\boldsymbol{k}}^{(2)}$ and $h_{2,\boldsymbol{k}}^{(2)}$ vanish because the corresponding integrands are odd in ξ. Additionally, we find that the $h_{2,\boldsymbol{k}}$ term included in $h_{1,\boldsymbol{k}}^{(3)}$ can be neglected in collisional plasmas, since it enters only at $\mathcal{O}(\delta^3)$ for $\omega/\nu \sim \delta \ll 1$. Thus, (51) is replaced by

$$\begin{aligned} h_{1,\boldsymbol{k}}^{(3)} &= -\frac{\mathrm{i}v}{\nu(v) - \mathrm{i}\omega_{\boldsymbol{k}}} \sum_{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}} \left(\frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}^{(1)}}{B_0} \cdot \boldsymbol{k}_2 \right) h_{0,\boldsymbol{k}_2}^{(2)} \\ &= \frac{v^3}{\nu(v) - \mathrm{i}\omega_{\boldsymbol{k}}} \frac{q f_0}{3T_0} \sum_{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}} \sum_{\boldsymbol{k}_2+\boldsymbol{k}_4=\boldsymbol{k}_2} \left(\frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_1}^{(1)}}{B_0} \cdot \boldsymbol{k}_2 \right) \left(\frac{\boldsymbol{B}_{\perp,\boldsymbol{k}_3}^{(1)}}{B_0} \cdot \boldsymbol{k}_4 \right) \frac{A_{\parallel,\boldsymbol{k}_4}^{(1)}}{\omega_{\boldsymbol{k}_2}} \frac{\omega_{\boldsymbol{k}_4} - \omega_{*,\boldsymbol{k}_4}^T(v)}{\nu(v) - \mathrm{i}\omega_{\boldsymbol{k}_4}}. \end{aligned} \tag{58}$$

... need to describe a detailed derivation process here...

Coupling $j_{\parallel,\boldsymbol{k}}$ obtained above with Ampere's law,

$$\mathrm{i}\left(\frac{\omega_{\boldsymbol{k}}}{\nu_c} - \frac{\omega_{*,n,\boldsymbol{k}}}{\nu_c} \left[1 + \frac{5}{2}\eta \right] + \mathrm{i}\frac{105}{16}\frac{\omega_{\boldsymbol{k}}}{\nu_c} \left[\frac{\omega_{\boldsymbol{k}}}{\nu_c} - \frac{\omega_{*,n,\boldsymbol{k}}}{\nu_c}(1+4\eta) \right] \right) \tag{59}$$

$$+ \frac{385}{32} \sum_{\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}} \left(\frac{\left| \boldsymbol{B}_{\perp,\boldsymbol{k}_1}^{(1)} \cdot \boldsymbol{k} v_{\mathrm{th}} \right|}{B_0 \omega_{\boldsymbol{k}_2}} \right)^2 \frac{\omega_{\boldsymbol{k}_2}}{\nu_c} \left(\frac{\omega_{\boldsymbol{k}}}{\nu_c} - \frac{\omega_{*,n,\boldsymbol{k}}}{\nu_c}(1+5\eta) \right) = \frac{3\pi}{32}(kd)^2, \tag{60}$$

where $\eta = L_{n_0}/L_{T_0}$. Note that we assume $\boldsymbol{k}_3 = -\boldsymbol{k}_1$, which means that...

...

3 Free energy in gyrokinetic system

Free energy of particles in a gyrokinetic framework is given by

$$K_s = \int \frac{T_{0s}|\delta f_s|^2}{2f_{0s}} \mathrm{d}\boldsymbol{v}. \tag{61}$$

We describe K_s in terms of fluid moments by means of the Legendre–Laguerre polynomial expansion.

3.1 Legendre polynomial expansion

Legendre polynomial is given by

$$P_l(\xi) = \frac{1}{2^l \xi!} \frac{\mathrm{d}^l}{\mathrm{d}\xi^l} (\xi^2 - 1)^l \tag{62}$$

where P_l is Legendre polynomial which satisfies the following differential equation and Bonnet's recursion:

$$\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi} P_l(\xi) = -l(l+1)P_l(\xi), \tag{63}$$

and

$$(l+1)P_{l+1} = (2l+1)\xi P_l - l P_{l-1}, \tag{64}$$

The first three lower P_l are given by

$$P_0(\xi) = 1, \tag{65}$$
$$P_1(\xi) = \xi, \tag{66}$$
$$P_2(\xi) = \frac{1}{2}(3\xi^2 - 1). \tag{67}$$

Orthogonality of Legendre polynomial is defined by

$$\int_{-1}^{1} P_l(\xi)P_m(\xi)\mathrm{d}\xi = \frac{2}{2l+1}\delta_{l,m}, \tag{68}$$

where $\delta_{l,m}$ denotes the Kronecker delta. Normalizing P_l by $2/(2l+1)$,

$$\int_{-1}^{1} \hat{P}_l(\xi)\hat{P}_m(\xi)\mathrm{d}\xi = \delta_{l,m}, \tag{69}$$

where

$$\hat{P}_l(\xi) = \sqrt{\frac{2l+1}{2}}\frac{1}{2^n\xi!}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}(\xi^2-1)^n, \tag{70}$$

and

$$\hat{P}_0(\xi) = \frac{1}{\sqrt{2}}, \tag{71}$$

$$\hat{P}_1(\xi) = \sqrt{\frac{3}{2}}\xi, \tag{72}$$

$$\hat{P}_2(\xi) = \frac{1}{2}\sqrt{\frac{5}{2}}(3\xi^2-1). \tag{73}$$

3.2 Associated Laguerre polynomial expansion

Associated Laguerre polynomial is given by

$$L_p^{(\alpha)}(x) = \mathrm{e}^x x^{-\alpha}\frac{1}{p!}\frac{\mathrm{d}^p}{\mathrm{d}x^p}(\mathrm{e}^{-x}x^{p+\alpha}) \tag{74}$$

The first three lower $L_p^{(\alpha)}(x)$ are given by

$$L_0^{(\alpha)}(x) = 1, \tag{75}$$

$$L_1^{(\alpha)}(x) = 1 - x + \alpha, \tag{76}$$

$$L_2^{(\alpha)}(x) = \frac{x^2 - 2(\alpha+2)x - (\alpha+1)(\alpha+2)}{2}. \tag{77}$$

Orthogonality of Laguerre polynomial is defined by

$$\int_0^\infty L_p(x)^{(\alpha)}L_q^{(\alpha)}(x)x^\alpha \mathrm{e}^{-x}\mathrm{d}x = \frac{\Gamma(p+\alpha+1)}{p!}\delta_{p,q}, \tag{78}$$

where $\delta_{p,q}$ denotes the Kronecker delta. Normalizing $L_p^{(\alpha)}(x)$ by $\Gamma(p+\alpha+1)/p!$,

$$\int_0^\infty \hat{L}_p(x)^{(\alpha)}\hat{L}_q^{(\alpha)}(x)x^\alpha \mathrm{e}^{-x}\mathrm{d}x = \delta_{p,q}, \tag{79}$$

where

$$\hat{L}_p^{(\alpha)}(x) = \sqrt{\frac{p!}{\Gamma(p+\alpha+1)}}\mathrm{e}^x x^{-\alpha}\frac{1}{p!}\frac{\mathrm{d}^p}{\mathrm{d}x^p}(\mathrm{e}^{-x}x^{p+\alpha}) \tag{80}$$

and

$$\hat{L}_0^{(\alpha)}(x) = \sqrt{\frac{1}{\Gamma(1+\alpha)}}, \tag{81}$$

$$\hat{L}_1^{(\alpha)}(x) = (1-x+\alpha)\sqrt{\frac{1}{\Gamma(2+\alpha)}}, \tag{82}$$

$$\hat{L}_2^{(\alpha)}(x) = \frac{x^2-2(\alpha+2)x-(\alpha+1)(\alpha+2)}{\sqrt{2}}\sqrt{\frac{1}{\Gamma(3+\alpha)}}. \tag{83}$$

3.3 Expansion of δf using Legendre-Laguerre polynomials

The perturbed distribution function obeying the gyrokinetic equation δf is expanded using Legendre-Laguerre polynomials as

$$\delta f(x,\xi) = f_0 \sum_{l,m} a_{l,m}\hat{P}_l(\xi)\hat{L}_m^{(\alpha)}(x)x^{l/2} \tag{84}$$

$$= \frac{n_0}{(\sqrt{\pi}v_{\mathrm{th}})^3}\sum_{l,m} a_{l,m}\hat{P}_l(\xi)\hat{L}_m^{(\alpha)}(x)\mathrm{e}^{-x}x^{l/2}, \tag{85}$$

where $\alpha = l+1/2$, $x = v^2/v_{\mathrm{th}}^2$, $\xi = v_\parallel/v$, $a_{l,m}$ is an expansion coefficient, respectively. Maxwellian represented by x is given by

$$f_0(x) = \frac{n_0}{(\sqrt{\pi}v_{\mathrm{th}})^3}\mathrm{e}^{-x}. \tag{86}$$

Multiplying $\hat{P}_{l'}(\xi)\hat{L}_n^{(\alpha)}(x)x^{l'/2}$ by (85), and integrating it over the velocity space ($\mathrm{d}\boldsymbol{v} = 2\pi v^2\mathrm{d}v\mathrm{d}\xi = v_{\mathrm{th}}^3\pi x^{1/2}\mathrm{d}x\mathrm{d}\xi$),

$$\int \delta f\hat{P}_{l'}(\xi)\hat{L}_n^{(\alpha)}(x)x^{l'/2}\mathrm{d}\boldsymbol{v} = \frac{n_0}{\sqrt{\pi}}\sum_{l,m} a_{l,m}\int_0^\infty \hat{L}_m^{(\alpha)}(x)\hat{L}_n^{(\alpha)}(x)\mathrm{e}^{-x}x^{(l+l'+1)/2}\mathrm{d}x\int_{-1}^{1}\hat{P}_l(\xi)\hat{P}_{l'}(\xi)\mathrm{d}\xi \tag{87}$$

$$= \frac{n_0}{\sqrt{\pi}}\sum_{l,m} a_{l,m}\int_0^\infty \hat{L}_m^{(\alpha)}(x)\hat{L}_n^{(\alpha)}(x)\mathrm{e}^{-x}x^{(l+l'+1)/2}\mathrm{d}x\,\delta_{l,l'} \tag{88}$$

$$\underbrace{=}_{l=l'} \frac{n_0}{\sqrt{\pi}}\sum_{m} a_{l',m}\,\delta_{m,n} \tag{89}$$

$$\underbrace{=}_{m=n} \frac{n_0}{\sqrt{\pi}}a_{l',n}. \tag{90}$$

Thus,

$$a_{l,m} = \frac{\sqrt{\pi}}{n_0}\int \delta f\hat{P}_l(\xi)\hat{L}_m^{(\alpha)}(x)x^{l/2}\mathrm{d}\boldsymbol{v}. \tag{91}$$

3.4 Relevant low-order fluid moments of δf

$$\tilde{n} = \int \delta f \mathrm{d}\boldsymbol{v} = \pi v_{\mathrm{th}}^3 \int x^{1/2} \delta f \mathrm{d}x \mathrm{d}\xi, \tag{92}$$

$$n_0 \tilde{u}_\parallel = \int v_\parallel \delta f \mathrm{d}\boldsymbol{v} = \pi v_{\mathrm{th}}^4 \int x\xi \delta f \mathrm{d}x \mathrm{d}\xi, \tag{93}$$

$$\frac{3}{2}\tilde{P} = \frac{m}{2}\int v^2 \delta f \mathrm{d}\boldsymbol{v} = \pi v_{\mathrm{th}}^3 T_0 \int x^{3/2} \delta \mathrm{d}x \mathrm{d}\xi, \tag{94}$$

$$\frac{3}{2} n_0 \tilde{T} = \pi v_{\mathrm{th}}^3 T_0 \int \left(x - \frac{3}{2}\right) x^{1/2} \delta f \mathrm{d}x \mathrm{d}\xi, \tag{95}$$

$$\tilde{p}_\perp = \frac{m}{2}\int v_\perp^2 \delta f \mathrm{d}\boldsymbol{v} = T_0 \int x(1-\xi^2)\delta \mathrm{d}\boldsymbol{v} = \pi v_{\mathrm{th}}^3 T_0 \int x^{3/2}(1-\xi^2)\delta f \mathrm{d}x \mathrm{d}\xi, \tag{96}$$

$$n_0 \tilde{T}_\perp = \pi v_{\mathrm{th}}^3 T_0 \int \left[x^{3/2}(1-\xi^2) - 1\right] x^{1/2} \delta f \mathrm{d}x \mathrm{d}\xi, \tag{97}$$

$$\tilde{p}_\parallel = m \int v_\parallel^2 \delta f \mathrm{d}\boldsymbol{v} = m \int v^2 \xi^2 \delta f \mathrm{d}\boldsymbol{v} = 2\pi v_{\mathrm{th}}^3 T_0 \int x^{3/2}\xi^2 \delta f \mathrm{d}x \mathrm{d}\xi, \tag{98}$$

$$n_0 \tilde{T}_\parallel = \pi v_{\mathrm{th}}^3 T_0 \int (2x\xi^2 - 1) x^{1/2} \delta f \mathrm{d}x \mathrm{d}\xi, \tag{99}$$

$$q_\parallel = \int \left(\frac{m}{2}v^2 - \frac{5}{2}T_0\right) v_\parallel \delta f \mathrm{d}\boldsymbol{v} = T_0 \int \left(\frac{v^2}{v_{\mathrm{th}}^2} - \frac{5}{2}\right) v_\parallel \delta f \mathrm{d}\boldsymbol{v} \tag{100}$$

We calculate the expansion coefficients given by (91) for each (l, m) using the relevant low-order fluid moments obtained above.

(i) $l = 0$ and $\alpha = l + 1/2 = 1/2$

$$a_{0,0} = \frac{\sqrt{\pi}}{n_0}\int \delta f \hat{P}_0(\xi)\hat{L}_0^{(1/2)}(x)\mathrm{d}\boldsymbol{v} = \frac{\sqrt{\pi}}{n_0}\hat{P}_0(\xi)\hat{L}_0^{(1/2)}(x)\pi v_{\mathrm{th}}^3 \int x^{1/2}\delta f \mathrm{d}x \mathrm{d}\xi = \frac{\pi^{1/4}}{n_0}\tilde{n}, \tag{101}$$

$$a_{0,1} = \frac{\sqrt{\pi}}{n_0}\int \delta f \hat{P}_0(\xi)\hat{L}_1^{(1/2)}(x)\mathrm{d}\boldsymbol{v} = \sqrt{\frac{2}{3}}\frac{\pi^{1/4}}{n_0}\pi v_{\mathrm{th}}^3 \int \left(\frac{3}{2} - x\right) x^{1/2}\delta f \mathrm{d}x \mathrm{d}\xi = -\sqrt{\frac{3}{2}}\pi^{1/4}\frac{\tilde{T}}{T_0}, \tag{102}$$

(ii) $l = 1$ and $\alpha = 3/2$

$$a_{1,0} = \frac{\sqrt{\pi}}{n_0}\int \delta f \hat{P}_1(\xi)\hat{L}_0^{(3/2)}(x) x^{1/2}\mathrm{d}\boldsymbol{v} = \frac{\sqrt{\pi}}{n_0}\frac{2}{\sqrt{3}\pi^{1/4}}\sqrt{\frac{3}{2}}\pi v_{\mathrm{th}}^3 \int \xi x \delta f \mathrm{d}x \mathrm{d}\xi = \sqrt{2}\pi^{1/4}\frac{\tilde{u}_\parallel}{v_{\mathrm{th}}}, \tag{103}$$

$$a_{1,1} = ... = -\sqrt{\frac{4}{5}}\pi^{1/4}\frac{q_\parallel}{n_0 T_0 v_{\mathrm{th}}}? \tag{104}$$

(iii) $l = 2$ and $\alpha = 5/4$

$$a_{2,0} = \frac{\sqrt{\pi}}{n_0}\int \delta f \hat{P}_2(\xi)\hat{L}_0^{(5/2)}(x)\, x\, \mathrm{d}\boldsymbol{v} = \frac{\pi^{1/4}}{\sqrt{3}n_0}\int (3\xi^2 - 1) x \delta f \mathrm{d}\boldsymbol{v} \tag{105}$$

$$= \frac{\pi^{1/4}}{\sqrt{3}n_0}\left(2\pi v_{\mathrm{th}}^3 \int x^{3/2}\xi^2 \delta f \mathrm{d}x \mathrm{d}\xi - \pi v_{\mathrm{th}}^3 \int x^{3/2}(1-\xi^2)\delta f \mathrm{d}x \mathrm{d}\xi\right) \tag{106}$$

$$= \frac{\pi^{1/4}}{\sqrt{3}}\frac{\tilde{p}_\parallel - \tilde{p}_\perp}{n_0 T_0} \tag{107}$$

$$= \frac{\pi^{1/4}}{\sqrt{3}}\frac{\tilde{T}_\parallel - \tilde{T}_\perp}{T_0} \tag{108}$$

where $(3\xi^2 - 1)x = (3v_\parallel^2 - v^2)/v_{\mathrm{th}}^2 = (2v_\parallel^2 - v_\perp^2)/v_{\mathrm{th}}^2 = 2x\xi^2 - x(1-\xi^2)$.

3.5 Free energy in terms of fluid moments

Finally, we describe free energy in the gyrokinetic system in terms of fluid moments. Substituting Legendre–Laguerre expansion (85) into (61),

$$K_s = \frac{T_{0s}}{2}\int f_{0s}\sum_{l,m}\sum_{l',m'} a_{l,m}a_{l',m'}\hat{P}_l(\xi)\hat{P}_{l'}(\xi)\hat{L}_m^{(\alpha)}(x)\hat{L}_{m'}^{(\alpha)}(x)x^{(l+l')/2}\mathrm{d}\boldsymbol{v} \tag{109}$$

$$= \frac{1}{2}\frac{n_{0s}T_{0s}}{\pi^{1/2}}\sum_{l,m}\sum_{l',m'} a_{l,m}a_{l',m'}\int \hat{P}_l(\xi)\hat{P}_{l'}(\xi)\mathrm{d}\xi\int \hat{L}_m^{(\alpha)}(x)\hat{L}_{m'}^{(\alpha)}(x)x^{(l+l'+1)/2}\mathrm{e}^{-x}\mathrm{d}x \tag{110}$$

$$= \frac{1}{2}\frac{n_{0s}T_{0s}}{\pi^{1/2}}\sum_{l,m}|a_{l,m}|^2. \tag{111}$$

Using the result of $a_{l,m}$ in (111),

$$K_s = \frac{1}{2}\frac{n_{0s}T_{0s}}{\pi^{1/2}}\left(|a_{0,0}|^2 + |a_{0,1}|^2 + |a_{1,0}|^2 + |a_{2,0}|^2 + \cdots\right) \tag{112}$$

$$= \frac{1}{2}n_{0s}T_{0s}\left|\frac{\tilde{n}_s}{n_{0s}}\right|^2 + \frac{3}{4}n_{0s}T_{0s}\left|\frac{\tilde{T}_s}{T_{0s}}\right|^2 + \frac{n_{0s}m_s}{2}|\tilde{u}_{\parallel,s}|^2 + \frac{n_{0s}T_{0s}}{6}\frac{|\tilde{T}_\parallel - \tilde{T}_\perp|^2}{T_{0s}^2}\cdots \tag{113}$$

4 Fluid–moment equations of δf

Our simulation results show that the energy associated with electron temperature fluctuations is dominant in both MTM and magnetic-shearless MTM. We derive the perturbed energy equation in terms of $\tilde{T}_\mathrm{e}$ to analyze the underlying physics in more detail.

The gyrokinetic equation for electrons considered here is given by

$$\frac{\partial h_\mathrm{e}}{\partial t} + \frac{1}{B_0}\{\phi, h_\mathrm{e}\} - \frac{v_\parallel}{B_0}\{A_\parallel, h_\mathrm{e}\} = \frac{q_\mathrm{e}f_{0\mathrm{e}}}{T_{0\mathrm{e}}}\left(\frac{\partial}{\partial t} + v_{*,\mathrm{e}}^T(v)\frac{\partial}{\partial y}\right)(\phi - v_\parallel A_\parallel) + C(h_\mathrm{e}), \tag{114}$$

$$h_\mathrm{e} = \delta f_\mathrm{e} + \frac{q_\mathrm{e}\phi}{T_{0\mathrm{e}}}f_{0\mathrm{e}}, \tag{115}$$

$$v_{*,\mathrm{e}}(v)^T = -\frac{T_{0\mathrm{e}}}{q_\mathrm{e}B_0}\left(\frac{1}{L_{n_{0\mathrm{e}}}} + \left[\frac{v^2}{v_{\mathrm{th,e}}^2} - \frac{3}{2}\right]\frac{1}{L_{T_{0\mathrm{e}}}}\right) \tag{116}$$

where $k_\perp\rho_\mathrm{e} \ll 1$ and the FLR for electrons is negligible. Note that the subscripts of variables are omitted hereafter .

4.1 Continuity equation

Taking the zeroth moment of (114) using (115), we obtain the continuity equation for δf.

$$\frac{\partial}{\partial t}\frac{\tilde{n}}{n_0} + \frac{1}{B_0}\left\{\phi, \frac{\tilde{n}}{n_0}\right\} - \frac{1}{B_0}\left\{A_\parallel, \tilde{u}_\parallel\right\} = v_{*,n}\frac{\partial}{\partial y}\frac{q\phi}{T_0}, \tag{117}$$

where $v_{*,n} = (-T_0/(qB_0))L_{n_0}^{-1}$.

4.1.1 Momentum equations

The $v_\parallel$–moment equation of (114) is considered here.

$$\frac{\partial}{\partial t}u_{\parallel,\mathrm{e}} + \frac{1}{B_0}\left\{\phi, u_{\parallel,\mathrm{e}}\right\} - \frac{v_{\mathrm{th,e}}^2}{2B_0}\left\{A_\parallel, \frac{q_\mathrm{e}\phi}{T_{0\mathrm{e}}} + \frac{\tilde{n}_\mathrm{e}}{n_{0\mathrm{e}}} + \frac{\tilde{T}_{\parallel,\mathrm{e}}}{T_{0\mathrm{e}}}\right\} = -\frac{q_\mathrm{e}}{m_\mathrm{e}}\left(\frac{\partial}{\partial t} + v_{*,n_{0\mathrm{e}}}(1+\eta_\mathrm{e})\frac{\partial}{\partial y}\right)A_\parallel + \mathcal{C}. \tag{118}$$

Consider $\partial A_\parallel/\partial t$ in the nonlinear phase, being given by

$$\frac{\partial A_\parallel}{\partial t} \sim -\frac{1}{B_0}\frac{m_\mathrm{e}}{q_\mathrm{e}}\left\{\phi, u_{\parallel,\mathrm{e}}\right\} + \frac{v_{\mathrm{th,e}}^2}{2B_0}\frac{m_\mathrm{e}}{q_\mathrm{e}}\left\{A_\parallel, \frac{q_\mathrm{e}\phi}{T_{0\mathrm{e}}} + \frac{\tilde{n}_\mathrm{e}}{n_{0\mathrm{e}}} + \frac{\tilde{T}_{\parallel,\mathrm{e}}}{T_{0\mathrm{e}}}\right\}. \tag{119}$$

(Memo: Normalizing its equation by the code unit,

$$\begin{aligned}\frac{\partial \hat{A}_\parallel}{\partial \hat{t}} &\sim -\frac{\hat{m}_\mathrm{e}}{\hat{q}_\mathrm{e}}\left\{\hat{\phi}, \hat{u}_{\parallel,\mathrm{e}}\right\} + \frac{\hat{v}_{\mathrm{th,e}}^2}{2}\frac{\hat{m}_\mathrm{e}}{\hat{q}_\mathrm{e}}\left\{\hat{A}_\parallel, \frac{\hat{q}_\mathrm{e}\hat{\phi}}{\hat{T}_{0\mathrm{e}}} + \frac{\hat{\tilde{n}}_\mathrm{e}}{\hat{n}_{0\mathrm{e}}} + \frac{\hat{\tilde{T}}_{\parallel,\mathrm{e}}}{\hat{T}_{0\mathrm{e}}}\right\} \\ &= \mathcal{N}^{\mathrm{ohm}}_{\phi,\tilde{u}_\parallel} + \mathcal{N}^{\mathrm{ohm}}_{A_\parallel,\phi} + \mathcal{N}^{\mathrm{ohm}}_{A_\parallel,\tilde{n}} + \mathcal{N}^{\mathrm{ohm}}_{A_\parallel,\tilde{T}_\parallel}\end{aligned} \tag{120}$$

)

$$\frac{1}{2}\frac{\partial |A_\parallel|^2}{\partial t} = -\frac{A_\parallel}{B_0}\left\{\phi, \frac{m_\mathrm{e}}{q_\mathrm{e}}u_{\parallel,\mathrm{e}}\right\} + \frac{T_{0\mathrm{e}}}{q_\mathrm{e}B_0}A_\parallel\left\{A_\parallel, \frac{q_\mathrm{e}\phi}{T_{0\mathrm{e}}} + \frac{\tilde{n}_\mathrm{e}}{n_{0\mathrm{e}}} + \frac{\tilde{T}_{\parallel,\mathrm{e}}}{T_{0\mathrm{e}}}\right\} - \frac{q_\mathrm{e}}{m_\mathrm{e}}\frac{v_{*,n_{0\mathrm{e}}}(1+\eta_\mathrm{e})}{2}\frac{\partial A_\parallel^2}{\partial y} + A_\parallel \mathcal{C}^{\mathrm{ohm}}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left\langle |A_{\parallel,k_y}|^2\right\rangle_x = \left\langle \mathcal{X}^{\mathrm{ohm}}_{\phi,\tilde{u}_\parallel,k_y}\right\rangle_x + \left\langle \mathcal{X}^{\mathrm{ohm}}_{A_\parallel,k_y}\right\rangle_x + \left\langle \mathcal{X}^{\mathrm{ohm}}_{A_\parallel,\tilde{n},k_y}\right\rangle_x + \left\langle \mathcal{X}^{\mathrm{ohm}}_{A_\parallel,\tilde{T},k_y}\right\rangle_x + \left\langle \mathcal{X}^{\mathrm{ohm}}_{\mathrm{coll}}\right\rangle_x, \tag{121}$$

where $\mathcal{X}^{\mathrm{ohm}}_{\boldsymbol{k}} \equiv \Re\left[A^*_{\parallel,\boldsymbol{k}}\mathcal{N}^{\mathrm{ohm}}_{\boldsymbol{k}}\right]$.

In the generalized Ohm's law, $\delta f = h$, thus, $\delta\Psi = \hat{\Psi}$. The temperature is isotropic in this system, being $T_\parallel = T_\perp = T$. Decomposing (119) into the Fourie-space,

$$\frac{\partial A_{\parallel,\boldsymbol{k}}}{\partial \hat{t}} \sim \mathcal{N}^{\mathrm{ohm}}_{\phi,\tilde{u}_\parallel,\boldsymbol{k}} + \mathcal{N}^{\mathrm{ohm}}_{A_\parallel,\phi,\boldsymbol{k}} + \mathcal{N}^{\mathrm{ohm}}_{A_\parallel,\tilde{n},\boldsymbol{k}} + \mathcal{N}^{\mathrm{ohm}}_{A_\parallel,\tilde{T},\boldsymbol{k}} \tag{122}$$

Multiplying (122) by $A^*_{\parallel,\boldsymbol{k}}$ to obtain the evolution equation of $|A_{\parallel,\boldsymbol{k}}|^2$, we get

$$\frac{1}{2}\frac{\partial |A_{\parallel,\boldsymbol{k}}|^2}{\partial \hat{t}} \sim \mathcal{X}^{\mathrm{ohm}}_{\phi,\tilde{u}_\parallel,\boldsymbol{k}} + \mathcal{X}^{\mathrm{ohm}}_{A_\parallel,\phi,\boldsymbol{k}} + \mathcal{X}^{\mathrm{ohm}}_{A_\parallel,\tilde{n},\boldsymbol{k}} + \mathcal{X}^{\mathrm{ohm}}_{A_\parallel,\tilde{T},\boldsymbol{k}}, \tag{123}$$

where $\mathcal{X}^{\mathrm{ohm}}_{\boldsymbol{k}} \equiv \Re\left[A^*_{\parallel,\boldsymbol{k}}\mathcal{N}^{\mathrm{ohm}}_{\boldsymbol{k}}\right]$. Summing (123) over k_x at a fixed k_y and dividing by $\sum_{k_x}|A_{\parallel,\boldsymbol{k}}|^2$ gives the rate

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\sum_{k_x}|A_{\parallel,\boldsymbol{k}}|^2\right)^{1/2} \equiv \gamma_{\mathrm{obs}}(k_y) \sim \gamma^{\mathrm{ohm}}_{\phi,\tilde{u}_\parallel}(k_y) + \gamma^{\mathrm{ohm}}_{A_\parallel,\phi}(k_y) + \gamma^{\mathrm{ohm}}_{A_\parallel,\tilde{n}}(k_y) + \gamma^{\mathrm{ohm}}_{A_\parallel,\tilde{T}}(k_y), \tag{124}$$

where

$$\frac{1}{2}\sum_{k_x}\frac{\partial |A_{\parallel,\boldsymbol{k}}|^2}{\partial \hat{t}}\bigg/\sum_{k_x}|A_{\parallel,\boldsymbol{k}}|^2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\hat{t}}\ln\left(\sum_{k_x}|A_{\parallel,\boldsymbol{k}}|^2\right) = \frac{\mathrm{d}}{\mathrm{d}\hat{t}}\ln\left(\sum_{k_x}|A_{\parallel,\boldsymbol{k}}|^2\right)^{1/2}, \tag{125}$$

and

$$\gamma^{\mathrm{ohm}}(k_y) \equiv \sum_{k_x}\mathcal{X}^{\mathrm{ohm}}_{\boldsymbol{k}}\bigg/\sum_{k_x}|A_{\parallel,\boldsymbol{k}}|^2. \tag{126}$$

4.1.2 Energy equation of Ohm's law

...

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{|\nabla_\perp A_\parallel|^2}{2\mu_0}+\frac{n_0 m}{2}|\tilde{u}_\parallel|^2\right)\mathrm{d}\boldsymbol{r}=\frac{n_0 T_0}{B_0}\left(\int\tilde{u}_\parallel\left\{A_\parallel,\frac{q\phi}{T_0}+\frac{\tilde{n}}{n_0}\right\}\mathrm{d}\boldsymbol{r}+\int\tilde{u}_\parallel\left\{A_\parallel,\frac{\tilde{T}}{T_0}\right\}\mathrm{d}\boldsymbol{r}\right)-D_{\mathrm{ohm}} \tag{127}$$

$$=\frac{n_0 T_0}{B_0}\left(\int\tilde{u}_\parallel\left\{A_\parallel,\frac{q\phi}{T_0}+\frac{\tilde{n}}{n_0}\right\}\mathrm{d}\boldsymbol{r}-\int\frac{\tilde{T}}{T_0}\left\{A_\parallel,\tilde{u}_\parallel\right\}\mathrm{d}\boldsymbol{r}\right)-D_{\mathrm{ohm}}, \tag{128}$$

where η term vanishes exactly.

4.2 Heat transport equation

4.2.1 Isotropic temperature $\tilde{T}$

Taking the $(1/2)mv^2$ moment of (114) using (115), we obtain the heat transport equation in terms of $\tilde{T}$ for δf.

$$\frac{3}{2}\frac{\partial}{\partial t}\left(\frac{\tilde{T}}{T_0}+\frac{\tilde{n}}{n_0}\right)+\frac{1}{B_0}\left\{\phi,\frac{3}{2}\frac{\tilde{n}}{n_0}\right\}+\frac{1}{B_0}\left\{\phi,\frac{3}{2}\frac{\tilde{T}}{T_0}\right\}-\frac{v_{\mathrm{th}}}{B_0}\left\{A_\parallel,\frac{\tilde{q}_\parallel}{n_0T_0v_{\mathrm{th}}}+\frac{5}{2}\frac{\tilde{u}_\parallel}{v_{\mathrm{th}}}\right\}$$
$$=\frac{3}{2}(v_{*,T}+v_{*,n})\frac{\partial}{\partial y}\frac{q\phi}{T_0}+\frac{1}{n_0}\int\frac{v^2}{v_{\mathrm{th}}^2}C(h)\mathrm{d}\boldsymbol{v}, \tag{129}$$

where $v_{*,T}=(-T_0/(qB_0))L_{T_0}^{-1}$ and

$$\tilde{q}_\parallel=\int\left(\frac{m}{2}v^2-\frac{5}{2}T_0\right)v_\parallel\delta f\mathrm{d}\boldsymbol{v}. \tag{130}$$

Substituting (117) into (129),

$$\frac{3}{2}\frac{\partial}{\partial t}\frac{\tilde{T}}{T_0}+\frac{1}{B_0}\left\{\phi,\frac{3}{2}\frac{\tilde{T}}{T_0}\right\}-\frac{v_{\mathrm{th}}}{B_0}\left\{A_\parallel,\frac{\tilde{q}_\parallel}{n_0T_0v_{\mathrm{th}}}+\frac{\tilde{u}_\parallel}{v_{\mathrm{th}}}\right\}=\frac{3}{2}v_{*,T}\frac{\partial}{\partial y}\frac{q\phi}{T_0}+\frac{1}{n_0}\int\frac{v^2}{v_{\mathrm{th}}^2}C(h)\mathrm{d}\boldsymbol{v}. \tag{131}$$

$$\frac{\partial}{\partial t}\left\langle\frac{\tilde{T}_{\mathrm{e}}}{T_{0\mathrm{e}}}\right\rangle_y=\frac{\partial}{\partial x}\left(\left\langle\frac{\tilde{T}_{\mathrm{e}}}{T_{0\mathrm{e}}}\frac{\partial}{\partial y}\frac{\phi}{B_0}\right\rangle_y-\frac{1}{3}\left\langle\frac{\tilde{q}_\parallel}{n_0T_0}\frac{\partial}{\partial y}\frac{A_\parallel}{B_0}\right\rangle_y-\frac{1}{3}\left\langle\tilde{u}_\parallel\frac{\partial}{\partial y}\frac{A_\parallel}{B_0}\right\rangle_y\right)+\left\langle\frac{1}{n_0}\int\frac{v^2}{v_{\mathrm{th}}^2}C(h)\mathrm{d}\boldsymbol{v}\right\rangle_y$$

$$\mathcal{Q}_{\phi,\tilde{T}}\equiv\left\langle\frac{\tilde{T}_{\mathrm{e}}}{T_{0\mathrm{e}}}\frac{\partial}{\partial y}\frac{\phi}{B_0}\right\rangle_y, \tag{132}$$

$$\mathcal{Q}_{A_\parallel,\tilde{q}_\parallel}\equiv-\frac{1}{3}\left\langle\frac{\tilde{q}_\parallel}{n_0T_0}\frac{\partial}{\partial y}\frac{A_\parallel}{B_0}\right\rangle_y, \tag{133}$$

$$\mathcal{Q}_{A_\parallel,u_\parallel}\equiv-\frac{1}{3}\left\langle\tilde{u}_\parallel\frac{\partial}{\partial y}\frac{A_\parallel}{B_0}\right\rangle_y \tag{134}$$

Introducing $\boldsymbol{E}\times\boldsymbol{B}$ flow $\boldsymbol{v}_{E\times B}$ and $\nabla_\parallel\equiv\boldsymbol{B}_\perp/B_0\cdot\nabla$,

$$\frac{3}{2}\frac{\partial}{\partial t}\frac{\tilde{T}}{T_0}+\frac{3}{2}\boldsymbol{v}_{E\times B}\cdot\nabla\frac{\tilde{T}}{T_0}+v_{\mathrm{th}}\nabla_\parallel\left(\frac{\tilde{q}_\parallel}{n_0T_0v_{\mathrm{th}}}+\frac{\tilde{u}_\parallel}{v_{\mathrm{th}}}\right)=\cdots. \tag{135}$$

(Memo: (132) normalized by the code unit is described as

$$\frac{3}{2}\frac{\partial}{\partial \hat{t}}\frac{\hat{\tilde{T}}}{\hat{T}_0}\epsilon t_0^{-1}+\frac{3}{4}\left\{\hat{\phi},\frac{\hat{\tilde{T}}}{\hat{T}_0}\right\}\epsilon t_0^{-1}-\frac{\hat{v_{\rm th}}}{2}\left\{\hat{A}_\parallel,\frac{\hat{\tilde{q}}_\parallel}{\hat{n}_0\hat{T}_0\hat{v_{\rm th}}}+\frac{\hat{\tilde{u}}_\parallel}{\hat{v_{\rm th}}}\right\}\epsilon t_0^{-1}=-\frac{3}{4}\frac{\hat{T}_0}{\hat{q}}\hat{L}_T^{-1}\frac{\partial}{\partial \hat{y}}\frac{\hat{q}\hat{\phi}}{\hat{T}_0}\epsilon t_0^{-1}+\hat{\mathcal{C}}\epsilon t_0^{-1}. \tag{136}$$

This equation holds for identically in ϵt_0^{-1}, thus,

$$\frac{3}{2}\frac{\partial}{\partial \hat{t}}\frac{\hat{\tilde{T}}}{\hat{T}_0}+\frac{3}{4}\left\{\hat{\phi},\frac{\hat{\tilde{T}}}{\hat{T}_0}\right\}-\frac{\hat{v_{\rm th}}}{2}\left\{\hat{A}_\parallel,\frac{\hat{\tilde{q}}_\parallel}{\hat{n}_0\hat{T}_0\hat{v_{\rm th}}}+\frac{\hat{\tilde{u}}_\parallel}{\hat{v_{\rm th}}}\right\}=-\frac{3}{4}\frac{\hat{T}_0}{\hat{q}}\hat{L}_T^{-1}\frac{\partial}{\partial \hat{y}}\frac{\hat{q}\hat{\phi}}{\hat{T}_0}+\hat{\mathcal{C}}. \tag{137}$$

)

Multiplying (132) by $\tilde{T}/T_0$ to obtain the energy density equation,

$$\begin{aligned}\frac{\partial}{\partial t}E_{\tilde{T}}(x,y,t) &= \frac{3}{2}\frac{\tilde{T}}{T_0}v_{*,T}\frac{\partial}{\partial y}\left(\frac{q\phi}{T_0}\right)-\frac{3}{2}\frac{\tilde{T}}{T_0}\boldsymbol{v}_{E\times B}\cdot\nabla\frac{\tilde{T}}{T_0}\\ &\quad -\frac{\tilde{T}}{T_0}v_{\rm th}\nabla_\parallel\left(\frac{\tilde{q}_\parallel}{n_0T_0v_{\rm th}}+\frac{\tilde{u}_\parallel}{v_{\rm th}}\right)+\frac{1}{n_0}\int\frac{v^2}{v_{\rm th}^2}\frac{\tilde{T}}{T_0}C(h)\mathrm{d}\boldsymbol{v} \qquad (138)\\ &= I_\phi+X_{\phi,\tilde{T}}+X_{A_\parallel,q_\parallel}+X_{A_\parallel,u_\parallel}-D_{\tilde{T}} \qquad (139)\end{aligned}$$

where

$$E_{\tilde{T}}=\frac{3}{4}\frac{\tilde{T}^2}{T_0^2}, \tag{140}$$

$$I_\phi=\frac{3}{2}\frac{\tilde{T}}{T_0}v_{*,T}\frac{\partial}{\partial y}\left(\frac{q\phi}{T_0}\right), \tag{141}$$

$$X_{\phi,\tilde{T}}=-\frac{\tilde{T}}{T_0}\frac{1}{B_0}\left\{\phi,\frac{3}{2}\frac{\tilde{T}}{T_0}\right\}=-\frac{3}{2}\frac{\tilde{T}}{T_0}\boldsymbol{v}_{E\times B}\cdot\nabla\frac{\tilde{T}}{T_0}, \tag{142}$$

$$X_{A_\parallel,q_\parallel}=\frac{\tilde{T}}{T_0}\frac{v_{\rm th}}{B_0}\left\{A_\parallel,\frac{\tilde{q}_\parallel}{n_0T_0v_{\rm th}}\right\}=-\frac{\tilde{T}}{T_0}v_{\rm th}\nabla_\parallel\frac{\tilde{q}_\parallel}{n_0T_0v_{\rm th}}, \tag{143}$$

$$X_{A_\parallel,u_\parallel}=\frac{\tilde{T}}{T_0}\frac{v_{\rm th}}{B_0}\left\{A_\parallel,\frac{\tilde{u}_\parallel}{v_{\rm th}}\right\}=-\frac{\tilde{T}}{T_0}v_{\rm th}\nabla_\parallel\frac{\tilde{u}_\parallel}{v_{\rm th}}, \tag{144}$$

$$D_{\tilde{T}}=-\frac{1}{n_0}\int\frac{v^2}{v_{\rm th}^2}\frac{\tilde{T}}{T_0}C(h)\mathrm{d}\boldsymbol{v}. \tag{145}$$

A helpful decomposition of $X_{A_\parallel,q_\parallel}$ to discuss is given by the Leibniz rule as

$$\begin{aligned}\{A_\parallel,\tilde{T}q_\parallel\}&=\tilde{T}\{A_\parallel,q_\parallel\}+\{A_\parallel,\tilde{T}\}q_\parallel \qquad (146)\\ \Rightarrow\tilde{T}\{A_\parallel,q_\parallel\}&=\{A_\parallel,\tilde{T}q_\parallel\}-\{A_\parallel,\tilde{T}\}q_\parallel, \qquad (147)\end{aligned}$$

thus,

$$X_{A_\parallel,q_\parallel}=-\frac{\tilde{T}}{T_0}v_{\rm th}\nabla_\parallel\frac{\tilde{q}_\parallel}{n_0T_0v_{\rm th}}=-v_{\rm th}\nabla_\parallel\left(\frac{\tilde{T}}{T_0}\frac{\tilde{q}_\parallel}{n_0T_0v_{\rm th}}\right)+v_{\rm th}\frac{\tilde{q}_\parallel}{n_0T_0v_{\rm th}}\nabla_\parallel\left(\frac{\tilde{T}}{T_0}\right) \tag{148}$$

4.3 Energy of temperature

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{3}{4}n_0T_0\left|\frac{\tilde{T}}{T_0}\right|^2\mathrm{d}\boldsymbol{r}=\frac{n_0T_0}{B_0}\int\left(\frac{\tilde{T}}{T_0}\left\{A_\parallel,\frac{\tilde{q}_\parallel}{n_0T_0}\right\}+\frac{\tilde{T}}{T_0}\left\{A_\parallel,\tilde{u}_\parallel\right\}\right)\mathrm{d}\boldsymbol{r}+I_{\tilde{T}}-D_{\tilde{T}} \tag{149}$$

A Pitch angle scattering collision operator

The collision operator due to the pitch angle scattering, which is generally known as Lorentz collision operator, is given by

$$C_{\mathrm{L}}(h) = \frac{\nu(v)}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}h, \tag{150}$$

where $\xi = v_\parallel/v$ is the pitch angle variable, $\nu(v)$ is the pitch angle scattering frequency dependent of the velocity and $h = h(v,\xi)$ is the velocity distribution function. Lorentz collision operator considers only collisions of light particles with heavy particles, neglecting energy loss, that is, the particle energy is not conserved.

A.1 $v_\parallel^2$ moment of (150)

$v_\parallel^2$ moment is

$$I \equiv \int v_\parallel^2 C_{\mathrm{L}}(h)\mathrm{d}\boldsymbol{v} \tag{151}$$

$$= \int v^2\xi^2\frac{\nu}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}h\mathrm{d}\boldsymbol{v}. \tag{152}$$

Transforming the velocity coordinate $(v_\parallel, v_\perp)$ to (v,ξ) using Jacobian,

$$\mathrm{d}\boldsymbol{v} = 2\pi\mathrm{d}v_\parallel v_\perp \mathrm{d}v_\perp = 2\pi v_\perp |J(v,\xi)|\mathrm{d}v\mathrm{d}\xi = 2\pi v^2\mathrm{d}v\mathrm{d}\xi, \tag{153}$$

where $v_\perp = v\sqrt{1-\xi^2}$ and $|J(v,\xi)| = v/\sqrt{1-\xi^2}$.

Using Legendre polynomials expansion,

$$I = -2\pi\sum_l \frac{l(l+1)}{2}\int_0^\infty v^4\nu(v)h_l(v)\mathrm{d}v\int_{-1}^1 \xi^2 P_l(\xi)\mathrm{d}\xi \tag{154}$$

$$= -2\pi\sum_l \frac{l(l+1)}{2}\int_0^\infty v^4\nu(v)h_l(v)\mathrm{d}v\int_{-1}^1 \left(\frac{2}{3}P_2(\xi)P_n(\xi) + \frac{1}{3}P_0(\xi)P_n(\xi)\right)\mathrm{d}\xi. \tag{155}$$

$l = 0, 2$ survive due to the orthogonality of Legendre polynomials defined in (68), but if $l = 0$, $I = 0$. Therefore,

$$\int v_\parallel^2 C_{\mathrm{L}}(h)\mathrm{d}\boldsymbol{v} = -\frac{8}{5}\pi\int_0^\infty v^4\nu(v)h_2(v)\mathrm{d}v \tag{156}$$

If collisions between particles are sufficiently strong, h_l with higher l is negligible. In [Drake+, 1980], h_l for $l \geq 2$ is neglected in the collisional regime, thereby satisfying $\int v_\parallel^2 C_{\mathrm{L}}(h)\mathrm{d}\boldsymbol{v} = 0$.