Physical picture of the destabilization mechanism of tearing mode

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1 Preliminary

1.1 Setup

We consider the destabilization mechanism of the tearing mode in a 2D slab configuration $\{x, y : -\pi \leq x \leq \pi,\ -\pi \leq y \leq \pi\}$. We give the anti-parallel magnetic field in the x direction as follow:

$$B_y^{\mathrm{eq}}(x) = B_{y0}\sin(x/a), \tag{1}$$

where a is the shear length. For this equilibrium magnetic filed, if the small perturbed magnetic filed $\tilde{A}_z(y,t) \propto \cos(k_y y)$ with the wavenumber k_y in the x direction, the X-point is formed at $(x, y) = (0, \pm\pi)$ as shown in Fig. 1. The 'tilde' symbol denotes the small perturbed amount. We demonstrate the time diffentiation of the $\tilde{A}_z$ takes positive at the X-point. The time variation of the magnetic field is described by Faraday's law,

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E}, \tag{2}$$

where the electric field $\boldsymbol{E}$ can be written by the Ohm's law as

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta \boldsymbol{j} - \frac{\nabla p_{0\mathrm{e}}}{n_{0\mathrm{e}} q_{\mathrm{e}}}. \tag{3}$$

The z component of the electric field contributes to the tearing instability leading to the magnetic reconnection. To linearize the z components of Eq. (2) and Eq. (3), we give the following equilibrium and perturbed amounts.

$$\begin{aligned} A_z &= A_{z0} + \tilde{A}_z, & A_{z0} &\gg \tilde{A}_z, \\ E_z &= E_{z0} + \tilde{E}_z, & E_{z0} &= 0, \\ v_x &= v_{x0} + \tilde{v}_x, & v_{x0} &= 0, \\ v_y &= v_{y0} + \tilde{v}_y, & v_{y0} &\gg \tilde{v}_y. \end{aligned} \tag{4}$$

Taking up to the first order of the perturbation and substituting Eq. (3) to Eq. (2), we can obtain

$$\frac{\partial \tilde{A}_z(x,y,t)}{\partial t} = -\eta \tilde{j}_z + \tilde{v}_x B_y^{\mathrm{eq}} - v_{y0}\tilde{B}_x, \tag{5}$$

where $\tilde{j}_z = -\nabla_\perp^2 \tilde{A}_z/\mu_0$ is the Ampére's law, $B_y^{\mathrm{eq}} = -\partial_x A_{z0}$, $\tilde{B}_x = \partial_y \tilde{A}_z$ and $v_{y0} = v_{\mathrm{De}}$ is the electron diamagnetic drift velocity. If we do not give the background pressure gradient, we can neglect the third term of RHS in Eq. (5).

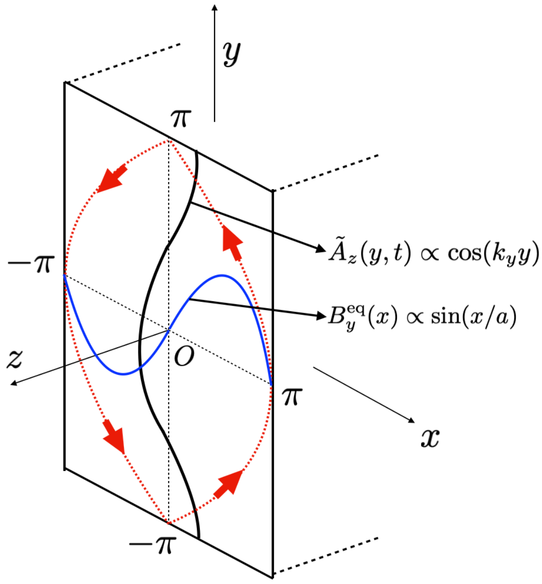

Figure 1: The image picture of the initial magnetic field configuration in the 2D slab. Blue, black and red solid lines are corresponding to the equilibrium, perturbed and composed magnetic fields, respectively. The X-point is formed at $(x, y) = (0, \pm\pi)$.

1.2 Derivation of the dominant equation

For generality, we give the small perturbed magnetic field as $\tilde{A}_z(x, y, t) = \bar{A}_z(x)\mathrm{e}^{\mathrm{i}(k_y y - \omega t)}$. Here $\omega = \omega_\mathrm{r} + \mathrm{i}\gamma$ is the complex mode frequency. Eq. (5) for the given perturbation can be written as

$$\left(\frac{\partial}{\partial t} + v_\mathrm{De}\frac{\partial}{\partial y}\right)\tilde{A}_z(x, y, t) = \frac{\eta}{\mu_0}\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)\tilde{A}_z(x, y, t) + \tilde{v}_x(x, y, t)B_y^{\mathrm{eq}}(x), \tag{6}$$

The first and second terms of RHS in Eq. (6) are corresponding to the diffusion and inflow at the X-point, respectively. In order the given perturbed magnetic field to be unstable, the profile of the $\tilde{v}_x$ plays an important role. We derive the profile with the fluid framework.

The incompressible 1-fluid MHD equation is given by

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \boldsymbol{j}\times\boldsymbol{B} - \nabla p, \tag{7}$$

$$\nabla\cdot\boldsymbol{v} = 0, \tag{8}$$

where, ρ and p are the mass density and pressure of fluid, respectively. The x and y components of the linearized Eq. (7) with Eq. (4) can be described by

$$\left(\frac{\partial}{\partial t} + v_\mathrm{De}\frac{\partial}{\partial y}\right)\tilde{v}_x = \frac{B_y^{\mathrm{eq}}}{\mu_0\rho}\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)\tilde{A}_z + \frac{B_y^{\mathrm{eq}\prime}}{\mu_0\rho}\frac{\partial}{\partial x}\tilde{A}_z - \frac{1}{\rho}\frac{\partial\tilde{p}}{\partial x}, \tag{9}$$

$$\left(\frac{\partial}{\partial t} + v_\mathrm{De}\frac{\partial}{\partial y}\right)\tilde{v}_y = \mathrm{i}\frac{B_y^{\mathrm{eq}\prime}}{\mu_0\rho}k_y\tilde{A}_z - \frac{\mathrm{i}k_y}{\rho}\tilde{p}, \tag{10}$$

The prime symbol denotes $\mathrm{d}/\mathrm{d}x$. We need the $\tilde{p}$ profile. Solving Eq. (10) with the incompressible condition $\tilde{v}_y = \mathrm{i}\partial_x \tilde{v}_x / k_y$ for $\tilde{p}$, the pressure can be derived as

$$\frac{\tilde{p}}{\rho} = \frac{B_y^{\mathrm{eq}\prime}}{\mu_0 \rho}\tilde{A}_z - \left(\frac{\partial}{\partial t} + v_{\mathrm{De}}\frac{\partial}{\partial y}\right)\frac{1}{k_y^2}\frac{\partial \tilde{v}_x}{\partial x}. \tag{11}$$

Substituting Eq. (11) to Eq. (9),

$$\left(\frac{\partial}{\partial t} + v_{\mathrm{De}}\frac{\partial}{\partial y}\right)\tilde{v}_x = \frac{B_y^{\mathrm{eq}}}{\mu_0 \rho}\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)\tilde{A}_z + \frac{B_y^{\mathrm{eq}\prime}}{\mu_0 \rho}\frac{\partial}{\partial x}\tilde{A}_z - \frac{B_y^{\mathrm{eq}\prime}}{\mu_0 \rho}\frac{\partial}{\partial x}\tilde{A}_z - \frac{B_y^{\mathrm{eq}\prime\prime}}{\mu_0 \rho}\tilde{A}_z + \left(\frac{\partial}{\partial t} + v_{\mathrm{De}}\frac{\partial}{\partial y}\right)\frac{1}{k_y^2}\frac{\partial^2 \tilde{v}_x}{\partial x^2} \tag{12}$$

$$\Leftrightarrow \left(\frac{\partial}{\partial t} + v_{\mathrm{De}}\frac{\partial}{\partial y}\right)\left(1 - \frac{1}{k_y^2}\frac{\partial^2}{\partial x^2}\right)\tilde{v}_x = \frac{B_y^{\mathrm{eq}}}{\mu_0 \rho}\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)\tilde{A}_z - \frac{B_y^{\mathrm{eq}\prime\prime}}{\mu_0 \rho}\tilde{A}_z \tag{13}$$

$$\Leftrightarrow \left(\frac{\partial}{\partial t} + v_{\mathrm{De}}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)\tilde{v}_x = -k_y^2 V_{\mathrm{A}}^2 \frac{B_y^{\mathrm{eq}}}{B_{y0}^2}\left[\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)\tilde{A}_z - \frac{B_y^{\mathrm{eq}\prime\prime}}{B_y^{\mathrm{eq}}}\tilde{A}_z\right], \tag{14}$$

where, $V_{\mathrm{A}}^2 = B_{y0}^2/(\mu_0 \rho)$ is the Alfvén speed defined by the reconnection magnetic filed.

The dominant equations are restated below. Understanding of the destabilization picture against the given perturbed magnetic field is equivalent to solving the following eigenvalue problem.

$$\frac{\mathrm{d}}{\mathrm{d}t'}\tilde{A}_z = \frac{\eta}{\mu_0}\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)\tilde{A}_z + \tilde{v}_x B_y^{\mathrm{eq}}, \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)\tilde{v}_x = -k_y^2 V_{\mathrm{A}}^2 \frac{B_y^{\mathrm{eq}}}{B_{y0}^2}\left[\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)\tilde{A}_z - \frac{B_y^{\mathrm{eq}\prime\prime}}{B_y^{\mathrm{eq}}}\tilde{A}_z\right]. \tag{16}$$

Here, $\mathrm{d}/\mathrm{d}t' \equiv \partial t + v_{\mathrm{De}}\partial y$.

2 Physical picture of destabilization mechanism

To strictly solve Eq. (15) and Eq. (16), we have to treat the singular magnetic filed structure in the x direction. The whole point of this discussion is to capture the physical picture of the destabilization mechanism of the tearing mode. For simplicity, the singularity in the x direction is represented by Δ' which is the solution of the outer region of Eq. (16). In the inner region which is the resistive layer, $\partial_x^2 \gg k_y^2$ and $B_y^{\mathrm{eq}} = B_{y0}x/a$ can be assumed. Thus, the eigenvalue equation can be simplified as

$$\frac{\mathrm{d}}{\mathrm{d}t'}\tilde{A}_z = \frac{\eta}{\mu_0}\frac{\partial^2}{\partial x^2}\tilde{A}_z + B_{y0}\frac{x}{a}\tilde{v}_x, \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t'}\frac{\partial^2}{\partial x^2}\tilde{v}_x = -\frac{1}{\tau_{\mathrm{A}}^2 B_{y0}}\frac{x}{a}\frac{\partial^2}{\partial x^2}\tilde{A}_z, \tag{18}$$

where, $1/\tau_{\mathrm{A}} = k_y V_{\mathrm{A}}$.

Tearing mode

We start to discuss the destabilization mechanism of the normal tearing mode, namely, we do not consider the background electron pressure gradient. Thus, $\mathrm{d}/\mathrm{d}t' \to \mathrm{d}/\mathrm{d}t$ and $\omega = \mathrm{i}\gamma$. We use the constant-$\psi$ approximation. Under the constant-ψ approximation, we define the following notation.

$$\tilde{A}_z(x,y,t) = \bar{A}_z \mathrm{e}^{\mathrm{i}(k_y y - \omega t)} = \left(\bar{A}_z(0) + A_z^1 x + \cdots\right) \mathrm{e}^{\mathrm{i}(k_y y - \omega t)} \approx \tilde{A}_z^0 + \tilde{A}_z^1 x \tag{19}$$

As is widely known, the solutions for $\tilde{v}_x$ and $\tilde{A}_z$ for $x \to +\infty$ must be matched to the asymptotic forms

$$\tilde{v}_x = \bar{v}_x(x) \mathrm{e}^{\mathrm{i}(k_y y - \omega t)} \approx \bar{v}_x^\infty \mathrm{e}^{\mathrm{i}(k_y y - \omega t)} = \tilde{v}_x^\infty = \text{const.}, \tag{20}$$

$$\tilde{A}_z = \bar{A}_z(x) \mathrm{e}^{\mathrm{i}(k_y y - \omega t)} \approx \bar{A}_z^{\infty,0} \left(1 + \frac{\Delta'}{2} x\right) \mathrm{e}^{\mathrm{i}(k_y y - \omega t)} = \tilde{A}_z^{\infty,0} \left(1 + \frac{\Delta'}{2} x\right). \tag{21}$$

Here, $\bar{A}_z^{\infty,0}$ denotes the constant value at $x = 0$ for $\bar{A}_z(x)$ at $x \to \infty$. With this boundary condition, Eq. (17) can be written as

$$\gamma \tilde{A}_z^{\infty,0} \left(1 + \frac{\Delta'}{2} x\right) = B_{y0} x \tilde{v}_x^\infty, \tag{22}$$

$$\Longrightarrow \gamma \tilde{A}_z^{\infty,0} \frac{\Delta'}{2} = B_{y0} \tilde{v}_x^\infty, \tag{23}$$

$$\Longleftrightarrow \tilde{A}_z^{\infty,0} = \frac{2 B_{y0} \tilde{v}_x^\infty}{\gamma \Delta'}, \tag{24}$$

with the condition that the resistivity η for $x \to \infty$ can be neglected. Eq. (18) may be integrated from $x = 0$ to $x = \infty$ to yield the constraint

$$\int_0^\infty \frac{\partial^2}{\partial x^2} \tilde{v}_x dx = -\int_0^\infty \frac{1}{\gamma \tau_\mathrm{A}^2 B_{y0}} x \frac{\partial^2}{\partial x^2} \tilde{A}_z dx \tag{25}$$

$$= -\int_0^\infty \frac{1}{\gamma \tau_\mathrm{A}^2 B_{y0}} \frac{\partial}{\partial x} \left[x^2 \frac{\partial}{\partial x} \frac{\tilde{A}_z}{x} \right] dx, \tag{26}$$

$$\overset{\tilde{v}_x^\infty = \text{const.}}{\cancel{\left.\frac{\partial \tilde{v}_x}{\partial x}\right|_{x=\infty}}} - \left.\frac{\partial \tilde{v}_x}{\partial x}\right|_{x=0} = -\frac{1}{\gamma \tau_\mathrm{A}^2 B_{y0}} \left[x \frac{\partial \tilde{A}_z}{\partial x} - \tilde{A}_z \right]_{x=0}^{x=\infty} \tag{27}$$

$$= -\frac{1}{\gamma \tau_\mathrm{A}^2 B_{y0}} \left[x \frac{\partial \tilde{A}_z}{\partial x} - \left(\tilde{A}_z^0 - \tilde{A}_z^1 x\right) \right]_{x=0}^{x=\infty} \tag{28}$$

$$= -\frac{1}{\gamma \tau_\mathrm{A}^2 B_{y0}} \left(-\tilde{A}_z^{\infty,0} + \tilde{A}_z^0\right). \tag{29}$$

Therefore,

$$\left.\frac{\partial \tilde{v}_x}{\partial x}\right|_{x=0} = \frac{1}{\gamma \tau_\mathrm{A}^2 B_{y0}} \left(-\tilde{A}_z^{\infty,0} + \tilde{A}_z^0\right). \tag{30}$$

Using Eq. (24) into Eq. (30), and re-organizing, we can obtain

$$\tilde{v}_x^{\infty} = -\frac{\gamma^2 \tau_{\mathrm{A}}^2 \Delta'}{2} \left.\frac{\partial \tilde{v}_x}{\partial x}\right|_{x=0}. \tag{31}$$

Applying Eq. (19) to Eq. (17) for $x \to \infty$ and substituting Eq. (31), the time evolution equation for $\tilde{A}_z^1$ can be derived as

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}_z^1 = -\frac{B_{y0}\gamma^2 \tau_{\mathrm{A}}^2 \Delta'}{2} \left.\frac{\partial \tilde{v}_x}{\partial x}\right|_{x=0}. \tag{32}$$

The exact solution of $\bar{v}_x(x)$ for the constant-ψ approximation is known as

$$\bar{v}_x(x) = -\sqrt{\frac{\pi}{8}}\Gamma\left(\frac{3}{4}\right) x^{1/2} \left(I_{\frac{1}{4}}\left(\frac{x^2}{2}\right) - L_{\frac{1}{4}}\left(\frac{x^2}{2}\right)\right), \tag{33}$$

where I and L are the modified Bessel function and the modified Struve function. With this exact solution, $\mathrm{d}\tilde{v}_x/\mathrm{d}x$ for $x \to 0$ becomes

$$\left.\frac{\partial \tilde{v}_x}{\partial x}\right|_{x=0} = -\sqrt{\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}. \tag{34}$$

Solving Eq. (18) for $\partial_x^2 \tilde{A}_z$ and integrating the equation with this exact solution of $\tilde{v}_x$,

$$\frac{\partial \tilde{A}_z}{\partial x} = \tilde{A}_z^1 = -\gamma\tau_{\mathrm{A}}\frac{225}{512}\frac{\pi^2}{\sqrt{2}\Gamma(9/4)\Gamma(13/4)}, \tag{35}$$

for $x \to \infty$. From Eq. (34) and Eq. (35),

$$\left.\frac{\partial \tilde{v}_x}{\partial x}\right|_{x=0} = -\frac{\gamma\tau_{\mathrm{A}}}{\sqrt{\pi}}\tilde{A}_z^1, \tag{36}$$

and using this into Eq. (32), we obtain the following relation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}_z^1 = \frac{B_{y0}\gamma^3 \tau_{\mathrm{A}}^3 \Delta'}{2\sqrt{\pi}}\tilde{A}_z^1. \tag{37}$$

(See the Mathematica script for the detail calculations for from Eq. (33) to (37).)